Texts and Monographs in Physics

H.L. Cycon R.G. Froese W. Kirsch B. Simon

Schrödinger Operators

with Application to Quantum Mechanics and Global Geometry

Springer-Verlag

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With 2 Figures

Springer-Verlag
Berlin Heidelberg New York
London Paris Tokyo

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ISBN 3-540-16759-5 Springer-Verlag Berlin Heidelberg New York
ISBN 0-387-16759-5 Springer-Verlag New York Berlin Heidelberg

Library of Congress Cataloging-in-Publication Data. Schrödinger operators, with application to quantum mechanics and global geometry. (Texts and monographs in physics) Chapters 1–11 are revised notes taken from a summer course given in 1982 in Thurnau, West Germany by Barry Simon. Bibliography: p. Includes index. 1. Schrödinger operator. 2. Quantum theory. 3. Global differential geometry. I. Cycon, H.L. (Hans Ludwig), 1942-. II. Simon, Barry. III. Series. QC174.17.S6S37 1987 515.7'246 86-13953

Typesetting: ASCO Trade Typesetting Limited, Hongkong
Offset printing: Druckhaus Beltz, 6944 Hemsbach
Bookbinding: J. Schäffer GmbH & Co. KG, 6718 Grünstadt
2153/3150-543210

Preface

In the summer of 1982, I gave a course of lectures in a castle in the small town of Thurnau outside of Bayreuth, West Germany, whose university hosted the lecture series. The Summer School was supported by the Volkswagen foundation and organized by Professor C. Simader, assisted by Dr. H. Leinfelder. I am grateful to these institutions and individuals for making the school, and thus this monograph, possible.

About 40 students took part in a grueling schedule involving about 45 hours of lectures spread over eight days! My goal was to survey the theory of Schrödinger operators emphasizing recent results. While I would emphasize that one was not supposed to know all of Volumes 1 – 4 of Reed and Simon (as some of the students feared!), a strong grounding in basic functional analysis and some previous exposure to Schrödinger operators was useful to the students, and will be useful to the reader of this monograph.

Loosely speaking, Chaps. 1 – 11 of this monograph represent "notes" of those lectures taken by three of the "students" who were there. While the general organization does follow mine, I would emphasize that what follows is far from a transcription of my lectures. Even with 45 hours, many details had to be skipped, and quite often Cycon, Froese and Kirsch have had to flesh out some rather dry bones. Moreover, they have occasionally rearranged my arguments, replaced them with better ones and even corrected some mistakes!

Some results such as Lieb's theorem (Theorem 3.17) that were relevant to the material of the lectures but appeared during the preparation of the monograph have been included.

Chapter 11 of the lectures concerns some beautiful ideas of Witten reducing the Morse inequalities to the calculation of the asymptotics of eigenvalues of cleverly chosen Schrödinger operators (on manifolds) in the semiclassical limit. When I understood the supersymmetric proof of the Gauss-Bonnet-Chern theorem (essentially due to Patodi) in the summer of 1984, and, in particular, using Schrödinger operator ideas found a transparent approach to its analytic part, it seemed natural to combine it with Chap. 11, and so I wrote a twelfth chapter. Since I was aware that Chaps. 11 and 12 would likely be of interest to a wider class of readers with less of an analytic background, I have included in Chap. 12 some elementary material (mainly on Sobolev estimates) that have been freely used in earlier chapters.

Los Angeles, Fall 1986 *Barry Simon*

Contents

1. Self-Adjointness

Self-adjointness of Schrödinger operators has been a fundamental mathematical problem since the beginning of quantum mechanics. It is equivalent to the unique solvability of the time-dependent Schrödinger equation, and it plays a basic role in the foundations of quantum mechanics, since only self-adjoint operators can ben understood as quantum mechanical observables (in the sense of *von Neumann* [361]).

It is an extensive subject with a large literature (see e.g. [293, 107, 196]) and the references given there), and it has been considerably overworked. There are only a few open problems, the most famous being Jörgens' conjecture (see [293, p. 339; 71, 317]).

We will not go into an exhaustive overview, but rather pick out some subjects which seen to us to be worth emphasizing. We will begin with a short review of the basic perturbation theorems and then discuss two typical classes of perturbations. Then we will discuss Kato's inequality. Finally, using an idea of Kato, we give some details of the proof of the theorem of Leinfelder and Simader on singular magnetic fields.

1.1 Basic Perturbation Theorems

First, we give some definitions (see [293, p. 162] for a more detailed discussion). We denote by A and B, densely-defined linear operators in a Hilbert space H, and by $D(A)$ and $Q(A)$, the operator domain and form domain of A respectively.

Definition 1.1. Let A be self-adjoint. Then B is said to be *A-bounded* if and only if

(i) $D(A) \subseteq D(B)$
(ii) there are constants $a, b > 0$ such that

$$\|B\varphi\| \leq a\|A\varphi\| + b\|\varphi\| \quad \text{for} \quad \varphi \in D(A) \ . \tag{1.1}$$

The infimum of all such a is called the *A-bound* (or relative-bound) of B. There is an analogous notion for quadratic forms:

Definition 1.2. Let A be self-adjoint and bounded from below. Then a symmetric operator B is said to be *A-form bounded* if and only if

(i) $Q(A) \subseteq Q(B)$
(ii) there are constants a, b > 0 such that

$$|\langle\varphi, B\varphi\rangle| \leq a\langle\varphi, A\varphi\rangle + b\langle\varphi,\varphi\rangle \quad \text{for} \quad \varphi \in Q(A) .$$

The infimum of all such a is called the *A-form-bound* (relative form-bound) of B.

Note that the operators in the above definitions do not need to be self-adjoint or symmetric [196, p. 190, p. 319]. We require it here because later propositions will be easier to state or prove for the self-adjoint case.

A subspace in H is called a *core* for A if it is dense in $D(A)$ in the graph norm. It is called a *form core* if it is dense in $Q(A)$ in the form norm.

There is an elementary criterion for relative boundedness.

Proposition 1.3. (i) Assume A to be self-adjoint and $D(A) \subseteq D(B)$. Then B is A-bounded if and only if $B(A + \mathrm{i})^{-1}$ is bounded. The A-bound of B is equal to

$$\lim_{|\gamma|\to\infty} \|B(A + \mathrm{i}\gamma)^{-1}\| .$$

(ii) (form version). Assume A to be self-adjoint, bounded from below and $Q(A) \subseteq Q(B)$. Then B is A-form-bounded if and only if $(A + \mathrm{i})^{-1/2}B(A + \mathrm{i})^{-1/2}$ is bounded. The A-form-bound of B is equal to

$$\lim_{|\gamma|\to\infty} \|(A + \mathrm{i}\gamma)^{-1/2}B(A + \mathrm{i}\gamma)^{-1/2}\| .$$

The assertion (i) can easily be seen by replacing φ by $(A + \mathrm{i}\gamma)^{-1}\psi$ in (1.1) and observing that $\|B(A + \mathrm{i}\gamma)^{-1}\| \leq [a + (b/|\gamma|)]$. (ii) follows analogously. Note that there is an extension of this notion which we use occasionally: We say that B is *A-compact* if and only if $B(A + \mathrm{i})^{-1}$ is compact. Here i can be replaced by any point of the resolvent set.

Now we will state the basic perturbation theorem which was proven by Kato over 30 years ago, and which works for most perturbations of practical interest.

Theorem 1.4 (Kato-Rellich). Suppose that A is self-adjoint, B is symmetric and A-bounded with A-bound $a < 1$. Then $A + B$ [which is defined on $D(A)$] is self adjoint, and any core for A is also a core for $A + B$.

We give a sketch of the proof. Note that self-adjointness of A is equivalent to $\mathrm{Ran}(A \pm \mathrm{i}\mu) = \mathrm{H}$ for some $\mu > 0$ [292, Theorem VIII.3]. Then, as above, we conclude from (1.1) that

$$\|B(A \pm \mathrm{i}\mu)^{-1}\| \leq a + \frac{b}{\mu} .$$

Thus, for μ large enough $C := B(A \pm \mathrm{i}\mu)^{-1}$ has norm less than 1, and this implies that $\mathrm{Ran}(1 + C) = \mathrm{H}$. This, together with the equation

$$(1 + C)(A \pm i\mu)\varphi = (A + B \pm i\mu)\varphi \quad \varphi \in D(A)$$

and the self-adjointness of A, implies that $\mathrm{Ran}(A + B \pm i\mu) = \mathrm{H}$. The second part of the theorem is a simple consequence of (1.1).

There are various improvements due to *Kato* [196] and *Wüst* [371] for the case $a = 1$, but in fact all the perturbations one usually deals with in the theory of Schrödinger operators have relative bound 0.

There is also a form version of Theorem 1.4 (due to Kato, Lax, Lions, Milgram and Nelson):

Theorem 1.5 (KLMN). Suppose that A is self-adjoint and bounded from below and that B is symmetric and A-bounded with form-bound $a < 1$. Then

(i) the sum of the quadratic forms of A and B is a closed symmetric form on $Q(A)$ which is bounded from below.
(ii) There exists a unique self-adjoint operator associated with this form which we call the form sum of A and B.
(iii) Any form core for A is also a form core for $A + B$.

For a proof, see [293, Theorem X.17]. We will denote the form sum by $A \dot{+} B$ when we want to emphasize the form character of the sum, otherwise we will write $A + B$.

Note that in spite of the parallelism between operators and forms, there is a fundamental asymmetry. There are symmetric operators which are closed but not self-adjoint. But a closed form which is bounded from below is automatically the form of a unique, self-adjoint operator [196, Theorem VI.2.1]. The form analog of essential self-adjointness, however, does exist: a suitable set being a form core. If one defines something to be a closed quadratic form, it is automatic that the associated operator is self-adjoint—one knows nothing, however, about the operator domain or the form domain. It is therefore a nontrivial fact that a convenient set (e.g. C_0^∞) is a form core.

1.2 The Classes S_ν and K_ν

In this book, we will study the sum $-\Delta + V$ in virtually all cases. But occasionally we will also study $(-i\nabla + a)^2 + V$ as operators or forms in the Hilbert space $L^2(\mathbb{R}^\nu)$. Here V is a real-valued function on $\mathbb{R}^\nu$ describing the electrostatic potential, and a is a vector-valued function which describes the magnetic potential. We denote by H_0 the self-adjoint representation of $-\Delta$ in $L^2(\mathbb{R}^\nu)$. In reasonable cases, one can think of V as a perturbation of H_0. Physically, this is motivated by the uncertainty principle which allows the kinetic energy to control some singularities of V if they are not too severe. This phenomenon has no classical analog. This is also practical since the Laplacian has an explicit eigenfunction expansion and integral kernel, and one knows everything about operator cores, etc.

There are two classes of perturbations we will discuss here. The class S_ν, which is an (almost maximal) class of operator perturbations of H_0 and the class K_ν which is the form analog of S_ν. S_ν was introduced originally by *Stummel* [352], and has been discussed by several authors (see e.g. [308]).

Definition 1.6. Let V be a real-valued, measurable function on $\mathbb{R}^\nu$. We say that $V \in S_\nu$ if and only if

a) $\lim_{\alpha\downarrow 0}\left[\sup_x \int_{|x-y|\le\alpha} |x-y|^{4-\nu}|V(y)|^2 d^\nu y\right] = 0 \quad \text{if} \quad \nu > 4$

b) $\lim_{\alpha\downarrow 0}\left[\sup_x \int_{|x-y|\le\alpha} \ln(|x-y|)^{-1}|V(y)|^2 d^\nu y\right] = 0 \quad \text{if} \quad \nu = 4$

c) $\sup_x \int_{|x-y|\le 1} |V(y)|^2 d^\nu y < \infty \quad \text{if} \quad \nu \le 3$.

For the reader who is disturbed by the lack of symmetry in the above definition, we remark that for $\nu \le 3$,

$$\sup_x \int_{|x-y|\le 1} |V(y)|^2 d^\nu y < \infty$$

is equivalent to

$$\lim_{\alpha\downarrow 0}\left[\sup_x \int_{|x-y|\le\alpha} |x-y|^{4-\nu}|V(y)|^2 d^\nu y\right] = 0 \ .$$

We define a S_ν-norm on S_ν by

$$\|V\|_{S_\nu} := \sup_x \int_{|x-y|\le 1} K(x,y;\nu)|V(y)|^2 d^\nu y \ ,$$

where K is the kernel in the above definition of S_ν. We now state (and prove) a theorem which shows how these quantities arise naturally. We denote, by $\|\cdot\|_{p,q}$, the operator norm for operators from $L^p(\mathbb{R}^\nu)$ to $L^q(\mathbb{R}^\nu)$, and by $\|\cdot\|_p$ the norm in $L^p(\mathbb{R}^\nu)$.

Theorem 1.7. $V \in S_\nu$ if and only if

$$\lim_{E\to\infty} \|(H_0 + E)^{-2}|V|^2\|_{\infty,\infty} = 0 \ . \qquad (1.2)$$

Proof. As with all functions of H_0, $(H_0 + E)^{-2}$ is a convolution operator with an explicit kernel $Q(x-y,E)$ [293, Theorem IX.29]. It has the following properties (see [308, Theorem 3.1, Chap. 6]).

1. $Q(x-y,E) \geq 0$,

2. $$Q(x-y,E) = \begin{cases} 0(|x-y|^{4-\nu}) & \text{if} \quad \nu > 4 \\ 0(\ln|x-y|^{-1}) & \text{if} \quad \nu = 4 \\ C & \text{if} \quad \nu \leq 3 \end{cases} \quad \text{as} \quad |x-y| \to 0 \ ,$$

3. $\sup\limits_{|x-y|>\delta} e^{|x-y|} Q(x-y,E) \to 0$ as $E \to \infty$, for any $\delta > 0$.

Using the elementary fact that

$$\sup_x \int_{|x-y|\leq 1} |V(y)|^2 \, dy < \infty$$

for any $V \in S_\nu$, it is not hard to see that $V \in S_\nu$ if and only if $\sup_x \int Q(x-y,E)|V(y)|^2 d^\nu y \to 0$ as $E \to \infty$. This gives the result, since $Q(\cdot - y, E)|V(y)|^2$ is a positive integral kernel and $\|A\|_{\infty,\infty} = \|A1\|_\infty$ holds for any A with positive integral kernel. □

The above result has an L^2 consequence by a standard "duality and interpolation" argument:

Corollary 1.8. If $V \in S_\nu$, then

$$\|(H_0 + E)^{-1} V\|_{2,2} \to 0 \quad \text{as} \quad E \to \infty \ . \tag{1.3}$$

Proof. Let $V \in S_\nu$. Then it is enough to show that

$$\|(H_0 + E)^{-1}|V|\|_{2,2}^2 \leq \|(H_0 + E)^{-2}|V|^2\|_{\infty,\infty} \ , \tag{1.4}$$

since (1.3) follows then by Theorem 1.7. Assume for a moment that V is bounded, and consider the function

$$F(z) := |V|^{2z}(H_0 + E)^{-2}|V|^{2-2z} \quad z \in \mathbb{C} \ .$$

$F(z)$ is an operator-valued function which is L^1 and L^∞-bounded and analytic in the interior of the strip $\{z \in \mathbb{C} | \operatorname{Re} z \in [0,1]\}$. Thus, by the Stein interpolation theorem [293, Theorem IX.21] and, using that (by duality)

$$\|(H_0 + E)^{-2}|V|^2\|_{\infty,\infty} = \||V|^2(H_0 + E)^{-2}\|_{1,1} \ ,$$

we get

$$\||V|(H_0 + E)^{-2}|V|\|_{2,2} \leq \|(H_0 + E)^{-2}|V|\|_{\infty,\infty} \ .$$

Since

$$\||V|(H_0 + E)^{-2}|V|\|_{2,2} = \|(H_0 + E)^{-1}|V|\|_{2,2}^2 \ ,$$

(1.4) follows for bounded V's, and by an approximation argument, also for all $V \in S_\nu$. □

Remark. Note that Corollary 1.8 implies that if $V \in S_\nu$, then it is H_0-bounded with H_0-bound 0 by Proposition 1.3 (Proposition 1.3 has to be slightly modified for the semibounded case we are considering here).

One might think that since S_ν is telling us something about L^∞-bounds and L^∞ is "stronger" than L^2, there would be no way going from L^2-bounds to S_ν. So the following theorem is interesting.

Theorem 1.9. Suppose there are $a, b > 0$ and a δ with $0 < \delta < 1$ such that, for all $0 < \varepsilon < 1$ and all $\varphi \in D(H_0)$

$$\|V\varphi\|_2^2 \le \varepsilon \|H_0\varphi\|_2^2 + a\exp(b\varepsilon^{-\delta})\|\varphi\|_2^2 \ .$$

Then $V \in S_\nu$.

Proof. We just have to pick the right φ's. Fix $y \in \mathbb{R}^\nu$, $t \in \mathbb{R}^+$, and consider the integral kernel

$$\varphi(x) := \sqrt{\exp(-tH_0)(x,y)} \ .$$

Then, noting that $\|\varphi\|_2 = 1$ and (by scaling)

$$\|H_0\varphi\|_2 = ct^{-2} \quad \text{for suitable} \quad c > 0$$

we have

$$[\exp(-tH_0)|V|^2](y) \le c\varepsilon t^{-2} + a\exp(b\varepsilon^{-\delta}) \ . \tag{1.5}$$

Now, take $\varepsilon := (1 + |\ln t|)^{-\gamma}$, where $\gamma := 2/(1+\delta)$, and multiply (1.5) by $t\exp(-tE)$ for $E > 0$. Then the R.H.S. of (1.5) is integrable in t and its integral goes to zero as $E \to \infty$. Now if we use the identity

$$(H_0 + E)^{-2} = \int_0^\infty te^{-tH_0}e^{-tE}\,dt$$

we get (1.2), and therefore $V \in S_\nu$ by Theorem 1.7. □

The second class of potentials we are considering here is K_ν, which is the form analog of S_ν. This type of potentials was first introduced by *Kato* [193]. See also *Schechter* [308] for related classes. K_ν was studied in some detail by *Aizenman* and *Simon* [7], and *Simon* [334].

Definition 1.10. Let V be a real-valued measurable function on $\mathbb{R}^\nu$. We say that $V \in K_\nu$ if and only if

a) $\lim\limits_{\alpha \downarrow 0}\left[\sup\limits_x \int\limits_{|x-y|\le\alpha} |x-y|^{2-\nu}|V(y)|d^\nu y\right] = 0, \quad \text{if} \quad \nu > 2$

b) $\lim_{\alpha\downarrow 0}\left[\sup_x \int_{|x-y|\le\alpha} \ln|(x-y)|^{-1}|V(y)|d^\nu y\right] = 0, \quad \text{if} \quad \nu = 2$

c) $\sup_x \int_{|x-y|\le 1} |V(y)|d^\nu y < \infty, \quad \text{if} \quad \nu = 1$.

We also define a K_ν-norm by

$$\|V\|_{K_\nu} := \sup_x \int_{|x-y|\le 1} \tilde{K}(x,y;\nu)|V(y)|d^\nu y$$

where $\tilde{K}$ is the kernel in the above definition of K_ν. Then virtually everything goes through as before.

Theorem 1.11 [7]. $V\in K_\nu$ if and only if

$$\lim_{E\to\infty} \|(H_0+E)^{-1}|V|\|_{\infty,\infty} = 0 .$$

The proof is the same as in Theorem 1.7.

Theorem 1.12 [7]. Suppose there are $a, b > 0$ and a δ with $0 < \delta < 1$ such that, for all $0 < \varepsilon < 1$ and all $\varphi\in Q(H_0)$

$$\langle\varphi, |V|\varphi\rangle \le \varepsilon\langle\varphi, H_0\varphi\rangle + a\exp(b\varepsilon^{-\delta})\|\varphi\|_2^2 .$$

Then $V\in K_\nu$.

The proof is again like that in Theorem 1.9 above (see also [7, Theorem 4.9]).

Remarks. (1) Both of the classes S_ν and K_ν have some nice properties:

a) If $\mu\le\nu$, then $K_\mu\subseteq K_\nu$ and $S_\mu\subseteq S_\nu$. By these inclusions we mean the following. Suppose $W\in K_\mu$ (resp. S_μ), and there is a linear surjective map T: $\mathbb{R}^\nu\to\mathbb{R}^\mu$ and $V(x) := W(T(x))$. Then $V\in K_\nu$ (resp. S_ν). The canonical example to think of here is an N-body system with $\nu = N\mu$, where a point $x\in\mathbb{R}^\nu$ is thought of as an N-tuple of μ-dimensional vectors $x = \langle x_1, \ldots, x_N\rangle$ and $Tx := x_i - x_j$ for some $i, j\in\{1, \ldots, N\}$, $i\ne j$.

b) There are some L_p-estimates which tell you when a potential is in K_ν (resp. S_ν), i.e.

$$L^p_{\text{unif}} \subseteq S_\nu \quad \text{if} \quad \begin{cases} p > \frac{\nu}{2} & \text{for} \quad \nu \ge 4 \\ p = 2 & \text{for} \quad \nu < 4 \end{cases}$$

and

$$L^p_{\text{unif}} \subseteq K_\nu \quad \text{if} \quad \begin{cases} p > \frac{\nu}{2} & \text{for} \quad \nu \ge 2 \\ p = 2 & \text{for} \quad \nu < 2 \end{cases}$$

where

$$L^p_{\text{unif}} := \{V|\sup_x \int_{|x-y|\le 1} |V(y)|^p\, dy < \infty\} \ .$$

The proof is a straightforward application of Hölder's inequality (see [7, Proposition 4.3]).

(2) If $V \in K_\nu$, then V is H_0-form bounded with relative bound 0. This follows again analogously from Proposition 1.3(ii), Theorem 1.11 and a corollary analogous to Corollary 1.8.

The classes K_ν and S_ν, however, are not the "maximal" classes with respect to the perturbation theorems, that is, one just misses the "borderline cases." This can be seen in the following:

Example. (a) Let $\nu \ge 3$ and

$$V(x) := |x|^{-2} |\ln|x||^{-\delta} \ .$$

Then $V \in K_\nu$ if and only if $\delta > 1$, but V is H_0-form bounded with bound 0 if and only if $\delta > 0$.

(b) Let $\nu \ge 5$ and V as in (a). Then $V \in S_\nu$ if and only if $\delta > 1/2$ but it is H_0-bounded with bound 0 if and only if $\delta > 0$. (a) is a consequence of [7, Theorem 4.11] and general perturbation properties (see [293, Chap. X.2]). (b) has a similar proof.

Remark. The above example shows that it is false that S_ν is contained in K_ν.

1.3 Kato's Inequality and All That

We will now sketch a set of ideas which go back to *Kato* [193], and which were subsequently studied by *Simon* [322, 327] (see also *Hess, Schrader* and *Uhlenbrock* [163]).

Let us first consider a vector potential a (magnetic potential), and a scalar V (electric potential) satisfying

$$a \in L^2_{\text{loc}}(\mathbb{R}^\nu)^\nu$$
$$V \in L^1_{\text{loc}}(\mathbb{R}^\nu), \quad V \ge 0 \ , \tag{1.6}$$

Then the formal expression

$$\tau := (-i\nabla - a)^2 + V$$

is associated with a quadratic form $h_{\max}$ (called the maximal form) defined by

$$Q(h_{\max}) := \{\varphi \in L^2(\mathbb{R}^\nu) | (\nabla - ia)\varphi \in L^2(\mathbb{R}^\nu)^\nu,\ V^{1/2}\varphi \in L^2(\mathbb{R}^\nu)\}$$

and

$$h_{\max}(\varphi,\psi) := \sum_{j=1}^{\nu} \langle(\partial_j - \mathrm{i}a_j)\varphi, (\partial_j - \mathrm{i}a_j)\psi\rangle + \langle V^{1/2}\varphi, V^{1/2}\psi\rangle$$

for φ, $\psi \in Q(h_{\max})$; ($\partial_j := \partial/\partial x_j$). Note that $h_{\max}$ is a closed, positive form (since it is the sum of ($\nu + 1$) positive closed forms), and therefore there exists a self-adjoint, positive operator H associated with $h_{\max}$, with

$$Q(H) = Q(h_{\max}) \quad \text{and}$$

$$\langle H\varphi, \psi\rangle = h_{\max}(\varphi,\psi) \quad \text{for} \quad \varphi, \psi \in D(H) \quad [196] \ .$$

Note also that (1.6) are the weakest possible conditions for defining a (closable positive) quadratic form associated with τ on $C_0^\infty(\mathbb{R}^\nu)$. The closure of this form [which is the restriction of $h_{\max}$ to $C_0^\infty(\mathbb{R}^\nu)$] is called $h_{\min}$. Our first theorem now says that these two forms coincide. Thus, the self-adjoint operator associated with the formal expression τ is, in a sense, unique.

Theorem 1.13 [329, 195]. $C_0^\infty(\mathbb{R}^\nu)$ is a form core for H.

We give only a sketch of the proof (see [329]).

Step 1.

$$e^{-tH}: L^2(\mathbb{R}^\nu) \to L^\infty(\mathbb{R}^\nu), \quad t \in \mathbb{R}^+ \ . \tag{1.7}$$

We only need to show that

$$|e^{-tH}\varphi| \leq e^{-tH_0}|\varphi|, \quad \varphi \in L^2(\mathbb{R}^\nu) \tag{1.8}$$

(which is the semigroup version of Kato's inequality, sometimes also called Kato-Simon inequality or diamagnetic inequality; see [327]), since (1.7) follows from (1.8) by using Young's inequality and the fact that $\exp(-tH_0)$ is a convolution with an L^2-integral kernel.

We know that H is a form sum of $\nu + 1$ operators. Therefore, we can use a generalized version of Trotter's product formula (shown by *Kato* and *Masuda* [198]) and get

$$\exp(-tH) = s - \lim_{n\to\infty}\left[\exp\left(\frac{t}{n}D_1^2\right)\exp\left(\frac{t}{n}D_2^2\right)\ldots\exp\left(\frac{t}{n}D_\nu^2\right)\exp\left(-\frac{t}{n}V\right)\right]^n , \tag{1.9}$$

where

$$D_j := \partial_j - \mathrm{i}a_j, \quad j \in \{1, \ldots, \nu\} \ .$$

Now, let

$$\lambda_j(x) := \int_0^{x_j} a(x_1, \ldots, x_{j-1}, y, x_{j+1}, \ldots, x_\nu)\,dy \ .$$

Then [329]

$$-\mathrm{i}D_j = \mathrm{e}^{\mathrm{i}\lambda_j}(-\mathrm{i}\partial_j)\mathrm{e}^{-\mathrm{i}\lambda_j} \ .$$

(Note that, in a "physicist's language", this means that in one dimension, magnetic vector potentials can always be removed by a gauge transformation.) Therefore

$$\exp\left(\frac{t}{n}D_j^2\right) = \exp(i\lambda)_j \exp\left(\frac{t}{n}\partial_j^2\right)\exp(-\mathrm{i}\lambda_j) \ , \quad \text{so that}$$

$$|\exp(tD_j^2)\varphi| \le \exp(t\partial_j^2)|\varphi|, \quad \varphi \in L^2(\mathbb{R}^\nu) \ . \tag{1.10}$$

Now (1.8) follows from (1.10), (1.9) and $|\exp(-tV/n)| \le 1$.

Step 2. $L^\infty(\mathbb{R}^\nu) \cap Q(H)$ is a form core for H.

This follows from (1.7) and the fact that $\mathrm{Ran}[\exp(-tH)]$ is a form core for H by the sepectral theorem.

Step 3. $L^\infty_{\mathrm{comp}}(\mathbb{R}^\nu) \cap Q(H)$ is a form core for H [where $L^\infty_{\mathrm{comp}}(\mathbb{R}^\nu) := \{\varphi \in L^2(\mathbb{R}^\nu) | \varphi \in L^\infty(\mathbb{R}^\nu), \operatorname{supp}\varphi \text{ is compact}\}$].

This follows by a usual cut-off approximation argument, i.e. choose $\eta \in C_0^\infty(\mathbb{R}^\nu)$ with $\eta = 1$ near 0, then consider, for any $\varphi \in L^\infty \cap Q(H)$

$$\varphi_n(x) := \eta\left(\frac{x}{n}\right)\varphi(x) \quad (n \in \mathbb{N})$$

then $\varphi_n \to \varphi$, $(n \to \infty)$ in the form sense. Now the proof will be finished by

Step 4. $C_0^\infty(\mathbb{R}^\nu)$ is a form core for H.

This follows by a standard mollifier argument, i.e. choose $j \in C_0^\infty(\mathbb{R}^\nu)$ such that $\int j(x)d^\nu x = 1$; set $j_\varepsilon := \varepsilon^{-\nu}j(x/\varepsilon)$, then for $\varphi \in L^\infty_{\mathrm{comp}} \cap Q(H)$ $\varphi_\varepsilon := j_\varepsilon * \varphi \in C_0^\infty$ and $\varphi_\varepsilon \to \varphi$, $(\varepsilon \to 0)$ in the form sense. □

Note that in the last two steps, it is crucial that the approximated function is in L^∞.

The next theorem is also a well-known result [193].

Theorem 1.14. Let $V \ge 0$, $V \in L^2_{\mathrm{loc}}(\mathbb{R}^\nu)$ and $a = 0$. Then $H := H_0 + V$ is essentially self-adjoint on $C_0^\infty(\mathbb{R}^\nu)$, i.e. $C_0^\infty(\mathbb{R}^\nu)$ is an operator core for H, and its closure is the form sum.

The proof is exactly the same as in Theorem 1.13 (replacing form cores by operator cores and form domains by operator domains) with one additional step. Once one notices that $L^\infty(\mathbb{R}^\nu) \cap D(H)$ is an operator core for H one uses the formula

$$H(\eta\varphi) = \eta H\varphi + 2\nabla\eta \cdot \nabla\varphi - \varphi\Delta\eta \tag{1.11}$$