JOHNSON AND KIOKEMEISTER

CALCULUS
WITH ANALYTIC GEOMETRY

THIRD EDITION

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R. E. JOHNSON

UNIVERSITY OF NEW HAMPSHIRE

F. L. KIOKEMEISTER

MOUNT HOLYOKE COLLEGE

EXERCISES BY M. S. KLAMKIN

FORD SCIENTIFIC LABORATORY, DEARBORN, MICHIGAN

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PREFACE

THIS THIRD EDITION reflects the manifold changes that have taken place in secondary school and college mathematics curriculums during the past four years. On the one hand, the notation of set theory has been incorporated, and on the other hand, the text has been extended to include Green's theorem. Between these extremes many other changes have been made, as noted below.

As in the previous editions, an intuitive discussion precedes the strict mathematical formulation of a topic whenever feasible. It is our strong feeling that both intuition and rigor are essential to a proper understanding of mathematics. By its very nature, mathematics must be done rigorously. Nevertheless, almost everything in the calculus arose from the consideration of a geometric or physical problem.

The more important changes in the third edition are as follows. The parabola is now defined in Chapter 2, and is thus available for applications of the derivative and integral. Ellipses and hyperbolas are not studied until Chapter 12, after formal integration. Limits are defined in Chapter 4 by use of neighborhoods. One-sided and infinite limits are also introduced in this chapter. Integration is presented earlier now, in Chapter 7, and is motivated by the geometric concept of measure. Indefinite integration is brought in quickly and is used henceforth in the book. New topics in the second chapter on integration, Chapter 8, are arc length and Simpson's rule. Separate chapters are now devoted to exponential functions (Chapter 9) and trigonometric functions (Chapter 10). In Chapter 13 the boundedness properties of a continuous function are proved by use of the compactness of a closed interval. Curves are defined in Chapters 15 and 19 as mappings of an interval into a plane or space. Vectors are considered to be translations in these chapters. Three-dimensional spaces of points and vectors are discussed in Chapter 16. Directional derivatives and differentials now play an important role in Chapter 17, on derivatives of functions of several variables. Chapter 19, on line and surface integrals, is entirely new.

Although this is primarily a calculus book, enough algebra and analytic geometry have been included to make it practically self-contained in these respects. Determinants, which are used throughout the text, are briefly described in an appendix.

It is not anticipated that an instructor will discuss every section of the book in class; some sections may be left for the students to read and others may be omitted altogether. For some classes it might be appropriate not to dwell too long on the proofs of the limit theorems (Section 7, Chapter 6), on upper and lower integrals (Section 4, Chapter 7), on the definition of e (Section 3, Chapter 9), on the inverse of a function (Section 6, Chapter 10), on uniform continuity (Section 8, Chapter 13), and so on. Sections 7–13 of Chapter 19 may be omitted on the ground that they more properly belong in an advanced calculus course. Although infinite series appear rather early in the book (Chapter 14), they may be postponed until later without affecting the continuity of the book. A section on partial derivatives is given early (Section 7, Chapter 10) for students who encounter functions of several variables in their elementary physics and chemistry courses. Clearly this section may be omitted at the teacher's discretion.

An added feature of this third edition is that the exercises have been revised and augmented. Professor M. S. Klamkin of Ford Scientific Laboratory, Dearborn, Michigan has assumed the responsibility of assembling the exercises for this edition. At the end of almost every section, there is a routine set I and a more challenging set II of exercises. In addition, a set of oral exercises together with a routine set I and a more challenging set II of exercises are included at the end of each chapter. As in previous editions, answers to the odd-numbered exercises appear at the end of the book. We request that any questions concerning the exercises or their suggested solutions be addressed to Professor Klamkin.

We acknowledge with gratitude the help of Professor R. P. Goblirsch in revising Chapters 15–19, on multidimensional calculus. Lack of space forced us to stop short of Stokes' theorem, but the necessary groundwork is there for the enterprising teacher who wishes to forge ahead on his own.

Essentially all of the material in this book has been tested in the classroom by the authors and their colleagues, and it has been improved as a result of this experience. Many other users of the previous editions have also suggested improvements. We shall continue to welcome such suggestions in the future. To all the people who have helped so greatly in the formation of this book, we express our deep appreciation.

R. E. J.
F. L. K.

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CONTENTS

1

TOPICS FROM ALGEBRA

CALCULUS, like algebra, is a study of numbers and functions. However, unlike algebra, it is primarily concerned with limiting processes rather than with factoring and solving equations. Some of the basic algebraic concepts of use in the calculus will be presented in this first chapter.

1 NUMBERS

The number system of elementary calculus is called the system of real numbers. In advanced mathematics the complex number system plays a vital role. Since we are concerned with elementary calculus in this book, we shall restrict our attention almost exclusively to real numbers.

Included among the real numbers are the *integers*

$$\cdots, -5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5, \cdots$$

and the ratios of integers, called *rational numbers*. Thus each rational number has the form p/q, where p and q are integers with $q \neq 0$. Every integer p is also a rational number, since it can be expressed in the form $p/1$.

There are real numbers that are not rational numbers; such numbers are called *irrational numbers*. For example, $1 + \sqrt{3}$, $\sqrt[3]{2}$, and π are irrational numbers.

Complex numbers are numbers of the form $a + bi$, where a and b are real numbers and i is a (nonreal) number having the property that

$i^2 = -1$. If $b \neq 0$, the complex number $a + bi$ is also called an *imaginary number*. For example, the quadratic equation

$$x^2 - 2x + 5 = 0$$

has as its solutions the imaginary numbers $1 + 2i$ and $1 - 2i$.

Whenever we use the word *number* by itself, it is tacitly understood we mean *real number*.

In modern terminology the system of real numbers is called a *field* because it has operations of addition and multiplication with the following properties, which define the concept of a field.

P1. *Associative laws.* $a + (b + c) = (a + b) + c$ and $a(bc) = (ab)c$ for all numbers a, b, and c.

P2. *Commutative laws.* $a + b = b + a$ and $ab = ba$ for all numbers a and b.

P3. *Distributive law.* $a(b + c) = ab + ac$ for all numbers a, b, and c.

P4. *Identity elements.* There exist numbers 0 and 1 such that $a + 0 = a$ and $a \cdot 1 = a$ for every number a.

P5. *Inverse elements.* Each number a has a negative $-a$ such that $a + (-a) = 0$; and each nonzero number a has a reciprocal $1/a$ such that $a \cdot (1/a) = 1$.

As is commonly known, other operations of subtraction and division may be defined in terms of addition and multiplication.

Just as the system of real numbers is a field, so too is the system of rational numbers and the system of complex numbers. However, the system of integers is not a field because the reciprocal of an integer need not be an integer.

A property of any field of numbers is that if a and b are numbers such that $ab = 0$, then either $a = 0$ or $b = 0$. Conversely, if either $a = 0$ or $b = 0$, then $ab = 0$. These two statements may be combined into one:

$$ab = 0 \text{ if and only if } a = 0 \text{ or } b = 0.$$

An important property of the real number field is that it is *ordered;* i.e., the set of nonzero real numbers can be separated into two parts, one made up of the positive numbers and the other of the negative numbers. Thus every real number is either a positive number, zero, or a negative number.

Both the sum and the product of two positive numbers are positive numbers. Two nonzero numbers a and b either *agree in sign* (i.e., both are positive or both are negative) or *differ in sign* (i.e., one is positive and one is negative). If a and b agree in sign, ab and a/b are positive numbers, whereas if a and b differ in sign, ab and a/b are negative numbers.

The nonzero numbers a and $1/a$ always agree in sign; a and $-a$ always differ in sign.

If $a \neq b$, then either a or b is greater than the other number. We write $a > b$ if a is greater than b, and $a < b$ if a is less than b (that is, b is greater than a). The relations "is greater than" and "is less than" ($>$ and $<$) may be formally defined as follows:

1.1 **DEFINITION.** If a and b are real numbers, then
(i) $a > b$ if $a - b$ is a positive number,
(ii) $a < b$ if $a - b$ is a negative number.

A meaningful algebraic expression involving relations such as "$>$" and "$<$" is called an *inequality.*

According to **1.1**, if a is a positive number, $a - 0$ is positive and $a > 0$. Conversely, if $a > 0$, then $a = a - 0$ is a positive number. Therefore

the number a is positive if and only if $a > 0$,

and, similarly,

the number a is negative if and only if $a < 0$.

The following laws of inequalities will be useful for our study.

1.2 If $a > b$ and $b > c$, then $a > c$ (transitive law).

1.3 For any number c, $a > b$ if and only if $a + c > b + c$.

1.4 For any $c > 0$, $a > b$ if and only if $ac > bc$.

1.5 For any $c < 0$, $a > b$ if and only if $ac < bc$.

The distinction between **1.4** and **1.5** should be noted. Thus multiplication by a positive number maintains the direction of the inequality, whereas multiplication by a negative number reverses the direction of the inequality.

These laws may be proved by using the properties of positive and negative numbers stated above. We illustrate this fact by proving one of the laws.

Proof of **1.4**: If $a > b$ and $c > 0$, then both $a - b$ and c are positive numbers. Hence $(a - b)c$, which is equal to $ac - bc$, is a positive number, and $ac > bc$ according to **1.1.** Conversely, if $ac > bc$ and $c > 0$, then both $ac - bc$ and $1/c$ are positive numbers. Hence $(ac - bc)\cdot(1/c)$, or $a - b$, is a positive number and $a > b$.

Each of **1.2–1.5** may be read in a somewhat different manner by replacing $a > b$ by $b < a$, and so on. Thus **1.4** could be read as follows: for any $c > 0$, $b < a$ if and only if $bc < ac$.

We may write

$$a < x < b \qquad \text{or} \qquad b > x > a$$

if $a < x$ and $x < b$. In this case x is a number between a and b. In a continued inequality such as this, the inequality signs always have the same direction. Thus we never write $a < x > b$ or $a > x < b$.

Other useful relations are "greater than or equal to" and "less than or equal to" ($\geq$ and $\leq$), which are defined in an obvious way:

$$a \geq b \text{ if } a > b \text{ or } a = b,$$
$$a \leq b \text{ if } a < b \text{ or } a = b.$$

A continued inequality such as

$$a \leq x < b$$

indicates either that x is between a and b or that $x = a$.

2 SETS AND INTERVALS

We have already had occasion to use the word *set* in discussing sets of numbers. In mathematics a collection of objects is called a *set*. Thus we speak of the set of all integers, the set of all real numbers, the set of all circles in a plane, and the set of all solutions of a given equation. The objects in a set are called *elements* when we do not wish to be specific.

It is common practice to designate sets by letters such as A, C, or S. For example, the set of all integers is designated by Z (perhaps for the German word *Zahl*) in many present-day algebra books. We frequently designate the elements of a set by lower-case letters such as a, c, or x. If A and B are sets such that every element of A is also an element of B, then we call A a *subset* of B and say that A is *contained in* B. Two sets A and B are equal if and only if each is a subset of the other, in which case we write $A = B$.

If the elements of a set can be enumerated, then we indicate the set by listing the elements and enclosing them with braces, $\{\quad\}$. For example,

$$\{1, 3, 5, 7, 9\}$$

indicates the set consisting of the five integers 1, 3, 5, 7, and 9. Clearly $\{1, 5, 9\}$ is a subset of this set.

We often describe a set by listing a typical element and stating its properties. For example,

$$\{2n + 1 \mid n \text{ an integer}\}$$

is the set of all odd integers. In this example $2n + 1$ is a typical element