CONTEMPORARY INTERMEDIATE ALGEBRA

CHARLES J. MERCHANT
University of Arizona

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PREFACE

Intermediate algebra is designed to follow the first two years of high school mathematics, consisting of introductory algebra and plane geometry. In the latter the student is introduced to a more or less fully developed axiomatic system, in which all theorems are demonstrated to be logical conclusions of a set of axioms. In the course of this development the mechanism of elementary deductive logic is introduced, sometimes tacitly, sometimes explicitly. For many students this is their first and often, regrettably, their last contact with formal logic.

It is the purpose of this text to continue the development of mathematics along formal logical lines. To this end the first chapter is given over to a presentation of a minimal amount of logical equipment designed primarily to acquaint the student with the concept and symbolism of implication and equivalence. Although it is attractive in many ways, the truth-table approach to the subject of implication was not used because of its immediate involvement in the paradoxes of material implication ("a false proposition implies anything.") It was felt that these difficulties could be better dealt with if approached in a more leisurely fashion than time permits in the usual intermediate algebra course.

Following this, the essentials of set notation and theory are presented in a separate chapter. The axioms of the real number system and a discussion of the axiom of order conclude the first section, entitled "Foundations." The notation and concepts which are introduced in this section are used throughout the text, and, it is hoped, should materially aid the student who continues on to precalculus mathematics. The concept and notation of function are employed consistently.

Full emphasis is placed throughout on what is certainly one of the most important goals of the intermediate algebra curriculum, the development of adequate manipulative skills. Lack of these skills is a serious obstacle to all further mathematical development, and sufficient exercises and worked out examples are provided to enable the student to achieve considerable proficiency. A final chapter on numerical computation covers some interesting topics not usually included, but which are of considerable utility.

It is with great pleasure that I acknowledge the assistance I have received from a number of kind and helpful people in writing this book. First, I owe a real debt of gratitude to Professor Harvey Cohn, formerly Head and presently Professor of the Department of Mathematics at the University of Arizona. In addition to giving me his constant encouragement, he read parts of the manuscript and made some very important suggestions and comments for which I am most grateful. Next, I would like to acknowledge the assistance of Professor William Rice, of St. Petersburg Junior College, who read the entire manuscript twice, and whose many helpful and pertinent suggestions were adopted in almost every instance.

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Finally, my sincere thanks to Mrs. Sarah Oordt, for her beautiful and patient typing of the manuscript through its numerous revisions.

CHARLES J. MERCHANT

CONTENTS

1 LOGIC

1.1 ALGEBRA AND LOGIC

Much of algebra is concerned with problems like the following:

Example 1: Prove: that $(a + b)(a - b) = a^2 - b^2$.

ANSWER:

$$(a + b)(a - b) = a \cdot a - a \cdot b + b \cdot a - b \cdot b = a^2 - ab + ab - b^2 = a^2 - b^2$$

Example 2: Solve: $3x^2 + 2x - 1 = 0$.

ANSWER:

$$x = \tfrac{1}{3} \quad \text{or} \quad x = -1$$

Example 3: Simplify: $\{5q - [4p + 2q - (p - 3q)]\}$.

ANSWER:

$$-3p$$

The letters and symbols in these examples are simply a very convenient shorthand for words, and each of these examples could be rephrased without using symbols or letters at all. Example 3, for instance, could be written, "If, from a certain number, a second number is subtracted three times, and this result is then subtracted from four times the first number plus twice the second number, and if this result is then subtracted from five times the second number, the result is minus three times the first number."

These examples are all statements about numbers. Each one of them says that one statement about certain numbers is the same as another statement about the same numbers. In other words, one statement about numbers is to be manipulated into another, more useful, statement about numbers which is equivalent to the first statement.

A central problem of algebra is the manipulation of statements concerning numbers to achieve a desired result. This result may be the solution of an equation, the simplification of a complicated expression, or the demonstration of a relationship of more than ordinary interest between certain quantities—a **theorem.** The manipulation of a statement is the process of drawing a conclusion from the statement or, alternatively, of finding the implication of a statement. In fact, it may be said that much of algebra is concerned with the implications which may be drawn from statements about numbers.

The discipline concerned with the whole subject of implication is called **logic.** Logic is much broader than algebra, however, in that its subject matter is the entire field of statements about any subject, not numbers alone. The laws of logic are rules designed to ensure that any conclusion drawn from a statement or set of statements is completely equivalent to the original statement or statements, in whole or in part, and contains no implication not in the original statements. It is the function of logic to furnish rules which can be used to justify each step in a chain of reasoning and make certain the final result is correct.

1.2 LOGIC

A famous example of deductive reasoning is the following:

Example 1: All men are mortal.
Socrates is a man.
Therefore, Socrates is mortal.

This example is seen to consist of three statements of fact, in the form of simple declarative sentences. The first two statements are the **premises,** and the last statement, the truth of which follows from the first two, is the **conclusion.** The premises and the conclusion taken together are said to constitute an **argument,** or **chain of reasoning.** This particular form of argument is called a **syllogism.** This is not the only possible form of syllogism, nor is a syllogism the only possible form of argument, but it is a typical example of the whole process of logical reasoning. Its strength lies in the fact that it is purely "formal." Any argument of the form

Example 2: all y's are z's;
x is a y;
therefore, x is a z

would have to be "logically" correct, even if the words used in place of the x's, y's, and z's were nonsense. Thus,

Example 3: all smollets are ploigs;
a zipf is a smollet;
therefore, a zipf is a ploig

is *logically* correct, even though what has been asserted and concluded remains something of a mystery!

Consider the following argument:

Example 4: All ripe apples are purple.
All purple things are good to eat.
Therefore, all ripe apples are good to eat.

The first premise is false, the second is not only false but dangerously so, though the conclusion is generally accepted as true, and certainly follows logically from the premises. Is this because the two false statements "cancel each other out," so to speak? Not necessarily.

For consider the following argument:

Example 5: Philadelphia is in Minnesota.
Minnesota is in the United States.
Therefore, Philadelphia is in the United States.

The first premise is false, the second is true, the conclusion is true, and the conclusion follows logically from the premises.

What may one conclude from the foregoing examples? *If the premises of a logically correct argument are true, the conclusion will be true, but from the truth of a conclusion we may not necessarily infer the truth of the premises.*

This fact is widely misunderstood and is the basis of much faulty reasoning. It has very important consequences in mathematics.

EXERCISE 1.1

Use common sense to say what conclusions may be drawn from the following premises:

1. All mammals are warm-blooded.
   No fish is warm-blooded.
2. All solid rocks sink in water.
   Some pumice-stone floats on water.
3. All equilateral triangles are isosceles.
   Some isosceles triangles are right triangles.
4. All x's are y's.
   Some y's are z's.
5. All kangaroos are marsupials.
   All opossums are marsupials.

6. Some mammals can fly.
All flying creatures have hollow bones.

7. George Washington was born in New York.
New York is in North America.

8. No sheep is a carnivore.
All carnivores have teeth.

9. I will go out if it is not raining.
It is raining.

10. Little children like to play in the mud.
No-one who is not careless of his personal appearance likes to play in the mud.

Criticize the following examples of logical reasoning.

11. All horses are quadrupeds.
All quadrupeds are vertebrates.
Therefore, all horses are vertebrates.

12. All quadrupeds are vertebrates.
All horses are vertebrates.
Therefore, all quadrupeds are horses.

13. All bipeds are vertebrates.
No horse is invertebrate.
Therefore, no horse is a biped.

14. All equilateral triangles have two equals sides.
All isosceles triangles have two equals sides.
Conclusion 1: All equilateral triangles are isosceles.
Conclusion 2: All isosceles triangles are equilateral.

15. In valid reasoning, a false conclusion never follows from the use of a valid procedure.
If division by zero is permitted, it may be shown that $1 = 2$.
Therefore, division by zero is not a valid procedure.

16. A good athlete trains hard, gets plenty of rest, and eats proper food.
John trains hard, gets plenty of rest, and eats proper food.
Therefore, John is a good athlete.

17. All intelligent persons go to college.
No stupid person goes to college.
Therefore, everybody in college is intelligent.

18. All intelligent persons who can afford to go to college.
Robert could not afford to go to college.
Therefore, Robert is intelligent.

19. Most college athletes spend so much time training that their grades suffer.
William was an all-American tackle, but he barely squeaked through college.
Nobody will ever know whether William was bright or not.

20. A cannibal who says "I could learn to like you" is being ambiguous.
No one should be trusted unless he is unambiguous.
Therefore, a cannibal who says "I could learn to like you" should not be trusted.

1.3 TERMS AND RELATIONS

The basic unit of logic is the term, which is simply the thing being discussed. Terms may be concrete—"John Smith" or "the Statue of Liberty"—or abstract—as "goodness," "beauty," "immortality." They may be particular or general, real or mythological. The symbol for the term may be a word, such as "grass" or "health," or a descriptive phrase, such as "the boy who mows the lawn" or "the state of well-being." However described or symbolized, the term must be clearly delimited; there must be no doubt as the subject of discourse.

The term in itself does not *assert* anything, i.e., it makes no statement *about* the subject under discussion. The words which tell *about* terms are words which define *relationships* between terms. Thus,

Example 1:

Term	*Relationship*	*Term*
the grass	is	green;
John	loves	Mary;
all triangles	have	three sides.

There are two particular kinds of relationships which are very important in logic: transitive relationships and symmetrical relationships. If we let R stand for a relationship which holds between terms, say A and B, then $A\ R\ B$ would be a statement proposition which asserts that A bears the relation R to B. Then if, whenever we are told $A\ R\ B$ and $B\ R\ C$, we may conclude immediately $A\ R\ C$, R is said to be a **transitive** relationship.

Example 2: If x is greater than y, and y is greater than z, then x is greater than z. ($R =$ "greater than.")

If Cleveland is east of Chicago, and New York is east of Cleveland, then New York is east of Chicago. ($R =$ "east of.")

Many important relationships are transitive: "greater than," "less than," "to the left of," "to the right of," "earns more than," and so forth. On the other hand, many important relationships are not transitive: If John loves Mary, and Mary loves William, we may not infer that John loves William.

A relationship is said to be **symmetrical** if, given $A\ R\ B$, we may immediately conclude $B\ R\ A$. "Equals" is probably the most important of the symmetrical relationships: If $a = b$, then $b = a$. There are many other symmetrical relationships which we encounter continually: "Near to," "unequal to," and so forth, are symmetrical relationships.

1.4 PROPOSITIONS AND PROPOSITIONAL FUNCTIONS

The statement of a relationship between terms is called a **proposition,** and the proposition is the starting point for logic. A *simple* proposition states a relationship between two terms. Most of the simple propositions which we will deal with are of the type in which the relationship between terms is stated by means of a form of the verb "to be."

Example 1: All men are mortal.
No horses are carniverous.
Some triangles are isosceles.
Some ripe apples are not red.

Propositions which are not of this form, such as "Dogs chase cats," may always be rephrased so that they are in this form. In this example, the restatement of this proposition would be "All (dogs) are (creatures who chase cats.)" This often results in ungainly and awkward sentences, but the result actually clarifies the relationship between the terms.

DEFINITION: A proposition is a statement which is either true or false, but not both.

Example 2: All whales are mammals. (true)
All apples are red. (false)

Many statements in ordinary language do not fall in this category.

Example 3: "What is it?"
"The White House."
"Full steam ahead!"

are not statements of fact, but a question, an address, and a command, respectively.

There are also sentences which look like statements of fact, and have the form of statements of fact, but are really only apparently so. Thus,

Example 4: "The tree is green;"
"Christmas comes on Friday;"
"$x = 3$"

have the form of simple propositions, but are not truly so. In this first instance, if by "the tree" we mean a certain maple tree in Cleveland, Ohio, on a certain day in August, the statement is true, but if we meant the same tree on a certain day in February, the statement is false. In the second case, if for "Christmas" we put "Christmas, 1964," the statement is true, but if instead we put "Christmas, 1963," the statement is false. Finally, "$x = 3$" is true if for "x" we substitute "3," and false if for "x" we substitute "2." Sentences such as these, which have the form of propositions, but are neither true nor false until substitutions have been made in them to particularize them, are called **open sentences, propositional functions,** or **propositional forms.**

EXERCISE 1.2

Determine whether the following sentences are propositions, propositional functions, or neither. *Remember,* if a statement is either true or false, it is a proposition; if a statement can be made into a proposition by specifying a term more closely, it is a propositional function; otherwise, a statement is neither, but is something like a question, a command, or an address.

1. Two plus two equals four.
2. x plus 2 equals four.
3. Does two plus two equal four?
4. All even numbers are divisible by two.
5. All even numbers are divisible by three.
6. Algebra makes extensive use of symbols.