MATHEMATICS
THE MAN-MADE UNIVERSE

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The Man-made Universe

An Introduction to the Spirit of Mathematics

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W. H. FREEMAN AND COMPANY
SAN FRANCISCO AND LONDON

Printed in the United States of America

Library of Congress Catalog Card Number: 63-7786 (C6)

MATHEMATICS: The Man-made Universe

A Series of Books in Mathematics

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TO HADASSAH

PREFACE

We all find ourselves in a world we never made. Though we get used to the kitchen sink, we do not understand the atoms which compose it. The kitchen sink, like all the objects surrounding us, is a convenient abstraction.

Mathematics, on the other hand, is completely the work of man. Each theorem, each proof, is the product of the human mind. In mathematics all the cards can be put on the table. In this sense, mathematics is concrete, whereas the world is abstract.

This book exploits that concreteness to introduce the general reader to mathematics. The "general reader" might be either the college student or the high school student, whatever his special interest might be, or the curious adult. This book grew out of a college course designed primarily to give students in many fields an appreciation of the beauty, extent, and vitality of mathematics. I had searched several years for a suitable text, but those I found were either too advanced or too specialized.

The subjects, chosen from number theory, topology, set theory, geometry, algebra, and analysis, can be presented to the reader having little mathematical background (some chapters use only grammar school arithmetic). Each topic illustrates some significant idea and lends itself easily to experiments and problems.

The reader is advised to take advantage of the concrete nature of mathematics as he reads each theorem and proof; to take nothing on faith; to be suspicious and vigilant; to examine each step of the reasoning; and to take seriously such suggestions as "the reader may provide an example of his own" or "the reader should check this theorem for some special cases before going on to the proof." It would be wise to read this book with pencil and paper always at hand.

The exercises at the end of each chapter vary in difficulty; some are just routine checks, whereas others raise questions that no one has answered. They give the reader a chance to test and apply his understanding of the material. Many of the exercises offer either alternate proofs of theorems proved in the text or further results. Some point out relations to other chapters.

Using the Map and Guide (page ix), the reader or teacher may choose his own route through this book. The recommended route takes the chapters in order.

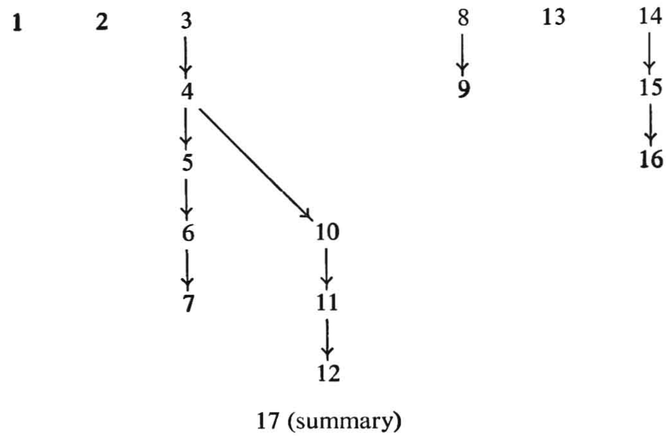
I would like to thank my students at the University of California at Davis for their comments on mimeographed versions of most of the chapters; my colleagues, Henry Alder and Curtis Fulton, for their encouragement and advice; and the artist, William Brown, who read several chapters and agreed that a proof can be beautiful.

It is a special pleasure to acknowledge the invaluable assistance of Robert Blair of Purdue University and George Raney of Wesleyan University, both of whom read the complete manuscript and made countless suggestions.

August 1962 SHERMAN K. STEIN

MAP AND GUIDE

In this map of the book "$a \longrightarrow b$" means that Chapter b depends in part on some material in Chapter a. The chapters are linked as follows.



Although, as the map indicates, the chapters need not be read in order, the recommended route does take the seventeen chapters in order. Many chapters refer for comparisons, contrasts, or exercises to earlier chapters. The map does not record these relations.

Chapters 1 and 2 introduce the mathematical style of thinking. Though referred to later, they are not used in the logical development of later chapters. Chapter 5 uses the main result of Chapter 4 (The Fundamental Theorem of Arithmetic). Chapter 11 uses results from Chapter 10 in the proof of Theorem 2. The proofs of Theorems 3 and 5 in Chapter 12 depend on Chapter 11. Chapter 15 uses the ordinary decimal representation of real numbers discussed in Chapter 14. Chapter 16 requires only Theorem 6 from Chapter 15. Chapter 17 is a general view and review.

Appendix A treats the four operations of arithmetic and algebra: addition, subtraction, multiplication, and division. The reader may use it simply as a reference, if and when needed, or he may choose to read it as a unit. It shows that the various manipulations met in arithmetic and algebra can all be based on eleven simple rules.

Appendix B treats the harmonic and geometric series, with an application of the latter to probability. This is a reference for Chapters 3, 14, and 15.

Appendix C defines spaces of any dimension. It is a reference for Chapter 15.

Throughout the text, E stands for exercise and R for reference. A starred reference presupposes more mathematical training than the general reader is expected to have; it is intended primarily as an aid to a teacher seeking background information. The contents of some references overlap.

Chapters 4, 7, 10, 11, 12, 13, 14, 15, and 16 use some high school algebra.

CONTENTS

Chapter 1

THE WEAVER

In this chapter and the next we introduce the reader to the mathematical way of thinking and, in particular, to the concepts of proof and theorem. All that we will need in these two chapters is the distinction between the *odd numbers*, beginning with one (1, 3, 5, ···) and the *even numbers*, beginning with zero (0, 2, 4, ···). From such a simple notion we will deduce important and surprising consequences. Indeed, the ancient duality of odd and even, which separates the *natural numbers* 0, 1, 2, 3, 4, 5, ··· into two types, will be of use several times in the course of this book; for example, in paths over highway systems, algebras, coloring of maps, and roots of polynomials—topics appearing in Chapters 8, 11, 13, and 15, respectively.

It is a puzzled weaver of hatbands who first introduces us to the mysteries of odd and even. To make his hatbands, this weaver braids several threads together, interchanging two at a time. Moreover, no two of his threads are of the same color. For instance, when he has only two threads, his pattern looks like

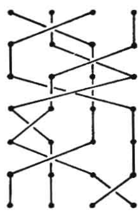
"After one switch," the weaver tells us, "neither thread is directly below its starting position. After the second switch, each is directly below its starting position. After three switches, each is again out of place; after four, they are back in place. Since I want these bands to go around a hat and not show the seam, my designs must have an even number of switches—at least if I have only two threads.

"Now, it seems to me that with more than two threads I should be able to make a seamless band with an odd number of switches. But, hard as I have tried, I have found no such design. Can you help me?"

Let us try to help the weaver find his design. Here is an experiment with three threads:

This diagram records the effect of four switches. Each switch interchanges two threads, as the weaver demands. The remaining thread is brought straight down (this is recorded by the vertical lines). Each of the three threads has been returned to its starting place, and so no seam will show. Regrettably, however, the number of switches is even; we did not find the weaver a design having an odd number of switches.

Let us try again. Here is a seamless design made with four threads:



The vertical lines record threads that are not switched; since there are four threads, two will not be switched at each stage.

The reader will notice that each of the four threads is back in place after six switches, again an even number. The reader is invited, at this point, to do some experimenting of his own with three, four, five, or more threads and to try designing a seamless hatband having an odd number of switches.

After several attempts we find ourselves in a disturbing position. We have found no design with an odd number of switches; we feel none can be found, but we are not sure of this. Perhaps there does exist a seamless design with very many threads and an odd number of switches.

What are we going to tell the weaver? He showed us why no two-thread seamless design having an odd number of switches could be found. It seems likely that if no designs exist at all—designs having any number of threads and an odd number of switches—then we ought to be able to explain why there are none. Experiments in mathematics prove nothing; they only point to a possible truth. Since no one can ever write down the endless list of all possible designs, we must somehow see into all designs at once and find a general principle that will tell us why the number of switches can never be odd.

We might find a clue to the more general question in the weaver's analysis of designs using only two threads. In order to simplify our diagrams, let us number his threads "1" and "2." The starting position is 1, 2. After one switch he has the backward position 2, 1; 2 is now to the left of 1. After another switch he has the original order 1, 2. The weaver's design of four switches we record simply as

1 2
2 1
1 2
2 1
1 2

We could translate the weaver's analysis into: after an odd number of switches, 1 and 2 are backward; after an even number of switches they are forward, in their usual order, 2 to the right of 1.

With this as a clue, let us look at our four-switch experiment with three threads, which we now record as

1 2 3
2 1 3
3 1 2
1 3 2
1 2 3

Looking only at threads 1 and 2, we notice that, unfortunately, they do not alternate "forward" and "backward" with each of the four switches. In fact, after each of the last three they are forward. Clearly, if there were more threads, then threads 1 and 2 might not be displaced at all. If we are to extend the weaver's analysis, we will have to pay attention to threads other than just 1 and 2.

In order to treat all the threads without favoritism, perhaps we should examine each pair of threads and see whether it is forward or backward. This might help; but then again it might not.

At the beginning position (position 1 2 3), each pair [(1, 2), (1, 3), (2, 3)] is forward; none is backward. At the next position, 2 1 3, the pair (2, 1) is backward, and (2, 3) and (1, 3) are forward. At 3 1 2 the pairs (3, 1) and (3, 2) are backward, and (1, 2) is forward. At 1 3 2 we have the one backward pair (3, 2), whereas (1, 3) and (1, 2) are forward. Finally, all are forward again.

The number of backward pairs for each arrangement runs successively through 0, 1, 2, 1 and returns to 0. With just two threads these numbers alternate 0, 1, 0, 1, and so on, as the weaver has told us. Clearly we do not have quite so simple a situation in our experiment with three threads.

Now let us look at our design with four threads and six switches, which we record as

1	2	3	4
3	2	1	4
3	4	1	2
2	4	1	3
4	2	1	3
1	2	4	3
1	2	3	4

For simplicity, let us call the number of backward pairs of an arrangement B; for example, B of 1, 2, 3, 4 is 0. As the reader may check, B for each of the seven arrangements is successively 0, 3, 4, 3, 4, 1, 0. For example, the third arrangement, 3, 4, 1, 2, has the backward pairs (3, 1), (3, 2), (4, 1), (4, 2). Thus B is 4, or we might write $B(3, 4, 1, 2) = 4$. Similarly, $B(1, 2, 3, 4) = 0$.

Now look at the two sequences we found; namely, 0, 1, 2, 1, 0 and 0, 3, 4, 3, 4, 1, 0. Naturally each begins and ends with 0. But, chaotic as they are, they do have something else in common: *each alternates even, odd, even, odd, and so on.* For two-thread designs the alternation is restricted simply to $0, 1, 0, 1, \cdots$, which is again even, odd, even, odd, $\cdots$. So we have a very promising clue for solving the weaver's problem. *All would be answered* if we could prove this statement:

If one switch is made in an arrangement of natural numbers, then the number of backward pairs always changes by an odd number.

We have reduced the question from one concerning weavers' designs that may involve billions of switches to one concerning the effect of a single typical switch. The thorough study of what happens when just one switch is made can be carried out with enough generality (if we make the bookkeeping sufficiently flexible) to cover completely the effect of any switch whatsoever.

We must scrutinize what happens to B when one switch is made in an arrangement. To be sure that our thinking will be valid for any switch in any arrangement, we will put hoods over all the natural numbers in question. This we can do quite easily by calling the left one of the two switched natural numbers "c" and the other "d." All we know is that c and d are two natural numbers; c may be less than d, or d may be less than c. The arrangement will look like

$$\_\_\_\_\_\_ c \_\_\_\_\_\_ d \_\_\_\_\_\_ ,$$

where the lines indicate that there may be more natural numbers in the arrange-