An Introduction to Differential Equations

Order and Chaos

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To my former, present, and future students

> Among all mathematical disciplines the theory of differential equations is the most important. . . . It furnishes the explanation of all those elementary manifestations of nature which involve time.
>
> MARIUS SOPHUS LIE
> Lobachevskii Prize, 1897

PREFACE

During the last decade of the 20th century, the teaching of calculus in North America has seen a split between the classical approach and a more intuitive one, commonly known as "reform." Some universities adopted the new point of view; others chose to stick with tradition. This split influenced the teaching of the sophomore differential equations course, which is a natural extension of the freshman calculus curriculum. The standard path in differential equations is based on exact techniques, whereas the modern one emphasizes qualitative methods. This textbook aims to reconcile these two directions and to offer the student a trip along all the main avenues of the field.

Goals

Our first goal was to strike a balance between these two points of view, keeping what is good and efficient from the classical approach, eliminating its nonessential and heavy-going aspects, and developing the qualitative theory as soon as possible. To present the student with a global and realistic image of the methods used today in differential equations, we cover all aspects of the theory: exact, qualitative, numerical, and computer techniques, as well as some elements of modeling.

Our second goal was to emphasize the usefulness of mathematics. Sophomores have reached the stage where their knowledge of algebra, geometry, and calculus bears fruit. They can apply the acquired tools to understand certain phenomena in fields ranging from physics, chemistry, business, and biology, to linguistics, literature, sports, and art. We have therefore gathered here several achievements of the theory, hoping to convince the students that mathematical thinking is a key component in modern society.

Our third goal was to make the textbook attractive and interesting. This was the most demanding aspect of our work. It involved intense library research, an effort of imagination, sparks of inventiveness, plus continuous refining and revision. Any writer knows how difficult it is to get the words right.

Structure

The textbook contains seven chapters. The first has only an introductory section in which we present the object of the theory and an overview of its methods. We also introduce some classification criteria, several equations and systems important in applications, and a few elements of modeling.

Chapter 2 deals with first-order equations. We present the exact methods for separable equations, discuss the variation of parameters for linear equations, and assign the homogeneous, exact, Bernoulli, and Riccati equations as problems. The qualitative methods involve drawing the slope field and the phase line, understanding the behavior of solutions near equilibria, and deciding about the existence and uniqueness of solutions for initial value problems. The method of successive approximations makes the passage from the qualitative to the numerical approach. We present the Euler and second-order Runge-Kutta methods as well as some rudiments of error theory. The computer techniques offer alternative ways of obtaining exact and numerical solutions and of drawing slope fields. At the end we invite the students to use their knowledge and imagination in doing some library research to come up with original ideas in modeling certain problems.

Chapter 3 presents the fundamental theory of linear second-order equations. We introduce the method of reduction, an algorithm for solving linear homogeneous equations with constant coefficients, the variation of parameters, and the method of undetermined coefficients. The qualitative approach reduces to the study of the phase plane for autonomous equations. We use the characteristic equation to establish the nature of the flow near an equilibrium and provide some elements of structural stability and bifurcation theory. The numerical methods generalize the ones presented in the previous chapter. The computer techniques give some new options for obtaining exact and numerical solutions and for drawing vector and direction fields. We end up with some modeling experiments.

Chapter 4 extends the results of Chapter 3 to linear systems. We introduce some elements of linear algebra, using a formalism that avoids operations with matrices. We present most results in terms of two- and three-dimensional systems, an approach that allows obvious generalization to n dimensions. The computer methods extend the ones of the previous chapter and also include programs for drawing the flow. These computer techniques can also be applied to nonlinear systems. We close the chapter with a few modeling experiments.

Chapter 5 deals with qualitative methods for nonlinear systems. We start with the linearization method and the study of the flow near equilibria based on the Hartman-Grobman theorem. We then investigate simple periodic orbits and cycles for two-dimensional systems in terms of polar coordinates, and we describe the connection between gradient and Hamiltonian systems and their reciprocal flows. We also introduce the notion of Liapunov stability, showing how the Liapunov function method can succeed when linearization fails. The chapter ends with a description of chaos in the language of symbolic dynamics and with some modeling experiments.

Chapter 6 covers differential equations and systems with the help of the Laplace transform. We first introduce the Laplace transform and its inverse and give an existence criterion. We then compute the Laplace transform of elementary and step functions and present a three-step method for solving differential equations and systems. The computer techniques can either apply this method directly or help with carrying it out step by step. We end the chapter with a few modeling experiments.

Chapter 7 considers exact and approximate power series solutions for differential equations. We first introduce a method that provides solutions near regular points and then use the Frobenius theorem to obtain solutions near regular singular points. The computer techniques implement these methods step by step. We end the chapter with some modeling experiments.

Applications

An important selection was the applications. To stress the importance of the theory of differential equations and to make the presentation attractive and interesting, we chose examples from various fields of human activity: anthropology, astronomy, population biology, brewing, business, chemistry, cooking, cosmology, rock climbing, ecology, economics, electronics, engineering, epidemiology, finance, mechanics, medicine, meteorology, oceanography, pharmaceuticals, physics, politics, space science, and sports. Some models are well established. Others are mere didactic toys.

We use differential equations to understand the motion of celestial bodies (pp. 6–9), model prices in a free-market economy (pp. 9–11, 196–197), compute the interest of investments (pp. 27–28), date the Shroud of Turin (pp. 28–29), cook a salmon (pp. 29–30), make a pharmaceutical drug (pp. 36–38), estimate the growth of the cougar population on Vancouver Island (pp. 46–47), follow the landing of Apollo 11 (pp. 47–48), study the swings of Galileo's pendulum (pp. 97–98, 256–258), and participate in some maglev transportation experiments (pp. 107–109). We also analyze the oscillations of water in a pipe (pp. 99–100), understand simple electric circuits (pp. 131–132), shed some light on the Tacoma Narrows Bridge disaster (pp. 116–119), determine the motion of a bungee jumper (pp. 137–139), and study the vibrations of a cantilever beam (pp. 139–141). We follow some chemical reactions in the search for an AIDS vaccine (pp. 183–184), determine the mixtures in a brewing technique (pp. 184–186), find the optimal shape of a rock-climbing tool (pp. 188–189), describe the evolution of two fish populations in the Tasmanian Sea (pp. 195–196), study epidemics with quarantine (pp. 203–204), and test the strength of buffer springs between the cars of trains (pp. 204–206). We further draw conclusions about the change of wolf and fox populations in northern Canada (pp. 232–233), explain the temperature variation of an engine and its coolant (pp. 239–240), see how lobsters scavenge (pp. 248–249), investigate why long-term weather forecasts are unreliable (pp. 260–261), determine the elasticity of a pole-vaulting pole (pp. 317–319), and study the escalation of expenditures in an arms race (pp. 319–320). Though far from displaying the entire spectrum of this theory, we can at least glimpse the variety of phenomena it describes.

Computers

A difficult choice was that of the computer environments. In a field like computer science, in which textbooks become obsolete soon after publication, opinions are changing fast, so it is impossible to satisfy everybody. In the end we decided to go for the three *M*'s: Maple, *Mathematica*, and MATLAB, which are popular in colleges and universities, have better chances of survival, and whose designers promote constructive upgrading. In Sections 2.7, 3.7, 4.6, 6.4, and 7.4 we present Maple, *Mathematica*, and MATLAB separately. This allows instructors a lot of freedom. They can teach one or all of them, treat them as independent entities or use them interactively while covering the other sections, or assign them as homework in connection with a computer project or modeling experiment.

Modeling

This is a difficult and time-consuming issue, which if stressed is done so at the expense of the core material on differential equations. Though we briefly discuss the modeling problem in most of our applications, we decided to emphasize this aspect at the end of every chapter and give instructors the option of assigning lab experiments to the students. Each modeling exercise has as a final goal the writing of an essay, which may contribute to the final grade. In Section 2.8 we deal with money investments, a model of the memory, the landing of Apollo 11, and a population dynamics experiment. Section 3.8 considers Galileo's pendulum, a model for bungee jumping, suspension bridges, and a simple electric circuit. In Section 4.7 we propose free-market models, a system describing malignant tumors and metastasis, another epidemic with quarantine, and a model for a decelerating train. Section 5.6 refers to chaotic aspects of the van der Pol, Düffing, and Lorenz equations, and to the three-body problem of celestial mechanics. In Section 6.5 we model car suspensions, electric circuits with ramped forcing, and instant shocks on harmonic oscillators. Finally, Section 7.5 deals with pole vault and arms race models and electric circuits with variable resistance and capacitance.

Historical Remarks

We mention names, dates, and nationalities for the mathematicians whose results we present, sometimes adding brief historical remarks. Some books use separate notes for this purpose. We chose to include the historical facts in the text in order to convey the feeling that mathematics is a cultural edifice built through collective human efforts.

Style

We tried to be direct and concise. Bombarded with information, students have no time and no desire to read more than they need. So we attempted to follow a geodesic toward our goal, keeping theory to a minimum. But we were generous with metaphors, figures, and examples, which give the text a friendly look, allowing a better and faster understanding. In most cases we took the route metaphor-theory-example, but sometimes we favored a more heuristic approach. We also aimed to strike a balance between rigor and intuition, relying on the latter whenever the former endangered clarity.

Teaching

As mentioned earlier, the text presents several methods—analytic, numerical, qualitative, and computer-based—emphasizing that each has its merits, depending on the circumstances. All these aspects can be covered in one term. But unless instructors have at their disposal 16 or 17 weeks, they will be unable to teach all the material. In the fall of 1998 and the spring of 1999, we covered Chapters 1 through 6 during the usual 13-week term at the University of Victoria, leaving aside the computer techniques, for which we used only 1 hour of demonstrations. Alternatively, in the summer of 1999, Cristina Stoica replaced Chapter 6 with Chapter 7. But there are many other choices an instructor can make with regard to chapters and even sections. For a more classical approach, Chapters 1, 2, 3, 4, 6, and 7 would be adequate. The computer sections can be intensively used or totally ignored. Somebody uninterested in numerical methods can simply avoid them. The only rule to follow is represented in the diagram below, which explains the logical construction of the textbook. Chapter 5, for example, can be taught only after going through Chapters 1, 2, 3, and 4. However, instructors have a lot of flexibility in what they can choose to teach and skip.

$$\begin{array}{ccccccccc} & & & & \mathbf{6} & & & & \\ & & & & \uparrow & & & & \\ \mathbf{1} & \to & \mathbf{2} & \to & \mathbf{3} & \to & \mathbf{4} & \to & \mathbf{5} \\ & & & & \downarrow & & & & \\ & & & & \mathbf{7} & & & & \end{array}$$

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CONTENTS

1 Introduction

Mathematics has, in modern times, brought [the idea of] order into greater and greater prominence.

BERTRAND RUSSELL
Nobel Prize for Literature, 1950

On November 7, 1940, some astounding pictures made the headlines of the North American television channels. They showed a man struggling to reach a car abandoned on a bridge that was wildly waving in the storm. After several unsuccessful attempts he gave up and, with visible efforts, returned to the shore. This proved a wise decision. The Tacoma Narrows Bridge near Seattle, a suspension structure more than a mile long, collapsed minutes later, witnessed by the helpless eyes of those who had dared to approach it. How could a construction of iron and concrete wave for days like a flag in the wind and then break all of a sudden? The answer is difficult (see Section 3.4), but some insight can be given through the theory of differential equations.

The history of science enumerates many achievements of this theory. The first is due to the English mathematician and physicist Isaac Newton (1642–1727), who in addition to being its cocreator, used it to show that the force that keeps the moon on its orbit is the same as the one that makes objects fall to the ground.

Shortly after this resounding success, the astronomer Edmond Halley (1656–1742), a friend of Newton, noticed the similarity of four cometary orbits observed in 1456, 1531, 1607, and 1682 and wondered whether they represent the same periodic trajectory. Using Newton's theory, Halley computed that the comet would return in 1758. He did not live to see the event, but his prediction proved accurate, and Halley's Comet has appeared in earth's skies on schedule three times more since then.

The discovery of the planet Neptune through numerical computations, performed independently by the French astronomer Jean Joseph Le Verrier (1811–1877) and the English astronomer John Couch Adams (1819–1892), was another significant success for the theory of differential equations. The observed orbit of Uranus had disagreed with the one predicted by theory. The two scientists argued that the discrepancy was due to the existence of some unknown planet. Using numerical methods, they computed the orbit of this hypothetical object, which was then observed on September 18, 1846.

First applied to the physical sciences, the theory of differential equations has later extended to other human activities ranging from engineering and biology to medicine, business, history, sports, and arts. The goal of this textbook is to introduce you to the main methods, ideas, techniques, and applications of this branch of mathematics, whose strength lies in its large applicability.

The Object of Study

A *differential equation* relates an unknown function and one or more of its derivatives. For example, the equation

$$x' = x \tag{1}$$

relates the function $x = x(t)$ and its derivative $x' = dx(t)/dt$. Unlike the unknowns of algebraic equations, which are numbers, the unknowns of differential equations are functions. Solving a differential equation means finding all its *solutions*, i.e., all functions that satisfy the equation. For example, $x(t) = e^t$ is a solution of equation (1) because $(e^t)' = e^t$. Can you find another solution?

The theory of differential equations has three main branches, which involve exact, numerical, and qualitative methods. Let us briefly describe them.

The *exact methods* are those meant to obtain all the solutions of a given equation. They first appeared more than three centuries ago at the same time as calculus. Though fundamental for understanding and developing further concepts, the exact methods have a narrow range of applications, because only a few classes of equations can be completely solved.

The *numerical methods* are designed to obtain, with some reasonable accuracy, particular solutions of a given equation. They have thrived during recent decades due to the invention of modern computers. Today these methods are widely used in practical problems ranging from physics and engineering to psychology and art, but, for reasons we will understand later, they offer good approximations only locally, i.e., on small intervals of the solution's domain. Therefore, in practical time-dependent problems, long-term predictions are difficult to achieve with this approach.

The *qualitative methods* are used to investigate properties of solutions without necessarily finding those solutions. For example, questions regarding existence and uniqueness, stability, or chaotic or asymptotic behavior can be answered with the help of these methods. Except for existence and uniqueness theorems, which appeared early in the development of the theory, the mainstream qualitative methods began to be developed toward the end of the 19th century, mainly through the work of the French mathematician Henri Poincaré. These methods are successful in understanding fundamental issues of the theory of differential equations.

We could add to this classification the process of *modeling*, which deals with obtaining the equations that describe certain phenomena and with interpreting the results of their analysis within the framework of the model. However, this is a much larger subject that goes beyond the theory of differential equations. We will present two examples of modeling later in this section and deal with this aspect in many of our applications.

The study of differential equations is a difficult task. Only a combination of quantitative, numerical, and qualitative methods brings insight toward understanding most problems. Mastering this theory at the research level requires knowledge in several branches of mathematics as well as a taste for applications. At the introductory level it asks for a solid background in mathematics, which includes the basic notions and techniques of calculus, algebra, and geometry.

Classification

There are several ways of classifying differential equations, of which we will consider here four criteria.

Domain of the unknown

The main distinction is between *ordinary differential equations* (ODEs) and *partial differential equations* (PDEs). ODEs involve functions of one variable and their derivatives, whereas PDEs concern functions of several variables and their partial derivatives.

EXAMPLE 1 The equations

$$x' = 2x^2, \tag{2}$$

$$u' = 2u - t^3, \tag{3}$$

$$v'' - 2tv' + v - 6 = 0 \tag{4}$$

are ODEs. For simplicity, the argument t of the functions x, u, and v is omitted. The equation

$$\frac{\partial u}{\partial t} = 2\left(\frac{\partial u}{\partial x}\right)^2 - 3xy\left(\frac{\partial u}{\partial y}\right)^3, \tag{5}$$

where the unknown function is $u(t, x, y)$, is a PDE. In this textbook we will deal only with ODEs.

Number of the unknowns

We distinguish between *single differential equations* and *systems of differential equations.*

EXAMPLE 2 All previous ODE examples have been single equations. The following ones are systems:

$$\begin{cases} x' = -y \\ y' = x, \end{cases} \tag{6}$$

$$\begin{cases} x_1' = x_1^2 + 5x_2^3 - x_3 \\ x_2' = \frac{1}{2}tx_1 - 3 \\ x_3' = x_1x_2x_3 \sin t. \end{cases} \tag{7}$$

The first system is two-dimensional and the second is three-dimensional.

Structure of the equation

We distinguish between *linear differential equations* and *nonlinear differential equations*. Linear equations are those whose left- and right-hand sides are linear functions (i.e., polynomials of degree 1) with respect to the unknown and its derivatives, whereas nonlinear ones do not satisfy this property. Linear systems are those formed by linear equations only, whereas nonlinear ones are those involving at least one nonlinear equation.

EXAMPLE 3 The equations

$$P' = t^2P, \tag{8}$$

$$u'' = (\sin\theta)u' + (\cos\theta)u \tag{9}$$

are linear in spite of having nonlinear coefficients (in t and θ, respectively). The equations

$$x' = 3x^2 + t, \tag{10}$$

$$y'' = -y' + yy' \tag{11}$$

are nonlinear. Also, system (6) in Example 2 is linear, whereas (7) is nonlinear.

Order of the equation

We say that an equation has *order k* if the highest derivative involved in the equation has order k.

EXAMPLE 4 The equations

$$w' = -4w + 3, \tag{12}$$

$$2X' - 5X'' = X + 7t^4, \tag{13}$$

$$\xi^2x''' = 6x' + x'' - 2\xi x \tag{14}$$

have order 1, 2, and 3, respectively. Systems (6) and (7) in Example 2 both have order 1.

Applications

The examples above have been chosen artificially, in the sense that they do not necessarily describe natural phenomena. But since the theory of differential equations is mainly concerned with those equations that have applications in other fields of human activity, we will present some examples from physics, astronomy, meteorology, chemistry, biology, anthropology, medicine, economics, and engineering. The area of applications is much larger than what we show here.

Some of the equations below are easy to solve, and we will solve them later; others continue to defy our attempts at obtaining explicit solutions. Progress in understanding them has been slow so far. Such differential equations can take the life-work of several generations of mathematicians and still remain poorly understood.

(i) The equation

$$x' = k \cdot x \tag{15}$$

models growth or decay problems. It says that, at a certain time, the rate at which a given quantity is changing is proportional to the amount existing at that time. This equation is used, for example, to determine the half-life of a radioactive substance or the doubling time of a money investment with continuously compounding interest. The well-known carbon dating method used in anthropology is based on this simple equation. In this case x represents the quantity of radioactive substance and k is a constant characteristic of the substance. The value of the constant can be determined through practical experiments and measurements. We will study this equation in detail in Section 2.2.

(ii) Newton's law of cooling or heating differs slightly from (15),

$$T' = k(T - \tau), \tag{16}$$

where k and τ are constants. It models the cooling or heating of a body immersed in a medium of constant temperature τ, where T is the unknown temperature function of the body, and k is a constant depending on the body. This equation describes a known physical phenomenon: that the rate at which the temperature of a body is changing is proportional to the difference between the temperatures of the medium and the body. We will study this equation in detail in Section 2.2.

(iii) The equation

$$u' = k(a - u)(b - u), \tag{17}$$

where k, a, and b are constants, models mixing problems in chemistry.

(iv) The logistic equation

$$p' = \lambda p(\alpha - p) \tag{18}$$

is a special case of equation (17), where α and λ are constants; it is used in biology and medicine for simple population and epidemiological models. We will study this equation in Section 2.4 in connection with a model that describes the evolution of the cougar population on Vancouver Island.

(v) The equation

$$x'' + bx' + kx = \gamma \sin \omega t, \tag{19}$$

where b, k, γ, and ω are constants, models a simple electric circuit or the motion of a damped spring with a periodic forcing term. It is connected to many practical problems, which range from bungee jumping to the collapse of the Tacoma Narrows Bridge. We will consider this equation in Section 3.4.