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Edited by A. Dold and B. Eckmann

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S. G. Krantz (Ed.)

Complex Analysis

Seminar, University Park PA, 1986

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PREFACE

This volume represents the proceedings of an intensive week of complex analysis at Penn State which was held during the week of March 10, 1986. The conference was attended by about fifteen people with similar interests, and every participant attended every lecture. The result was an enjoyable and rewarding exchange of ideas.

The lead article in this volume is a rather personal assessment of progress in Several Complex Variables in the past fifteen years. Subsequent articles in the volume point to a number of new paths which we expect the subject to follow. We hope that the volume will be especially helpful to students and new members in the field, as well as to people who are already established.

We are grateful to the Department of Mathematics and the College of Science at the Pennsylvania State University for funding this conference.

Steven G. Krantz
St. Louis, Missouri USA
March, 1987

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Recent Progress and Future Directions in Several Complex Variables

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Section 0 Introduction

I doubt that anyone is qualified to produce a list of the most important results in several complex variables from the last ten or fifteen years--certainly I am not. What I want to do here is to comment on a few areas with which I have some familiarity, and on which the talks given at this conference impinge. A lot has been written about the fallout from the construction of integral representations by Henkin, Ramirez, and others (see [HEN1] and [RAM]). And the significance and pervasiveness of the theory of the $\bar{\partial}$-Neumann problem and the weighted L^2 estimates of Hörmander seem to be well-known. Therefore I would rather concentrate here on the consequences of a few key events which occurred in the early 1970's. These are the following:

(i) The Kohn-Nirenberg example (1973)
(ii) The worm domain (Diederich and Fornaess, 1977)
(iii) Bounded strictly plurisubharmonic exhaustion functions (Diederich and Fornaess, 1977)
(iv) Points of finite type and subelliptic estimates for the $\bar{\partial}$-Neumann problem (Kohn, 1972)
(v) Fefferman's mapping theorem (1974)

It is my view that these discoveries, and related results by dozens of other mathematicians, have completely altered the way that we think about several complex variables and, in particular, they have changed the course of research.

The fundamental change that has occurred is simply this: much of the work prior to the early 1970's, and even the work that goes into the original proof of Fefferman's mapping theorem, entailed a very careful understanding of <u>strongly</u> pseudoconvex points. The Levi problem had been reduced to the strongly pseudoconvex case in the 1930's, and the solution of the Levi problem in the 1950's included a detailed analysis of the local geometry of the positive definite Levi form (see, for instance, [BER]). Likewise, Kohn's solution of the $\bar{\partial}$-Neumann problem (see [FOK]) hinged on a consideration of the Neumann boundary conditions at strongly pseudoconvex points. Fefferman's work in [FE1] required a much deeper analysis of the approximations of strongly pseudoconvex domains by balls. It is safe to say that Fefferman's work in [FE1], [FE2], [FE3], including work of Folland/Stein [FOS1] and others that it built on, represents the deepest understanding to date of strong pseudoconvexity.

What is very striking is that, until the early 1970's, no one had thought carefully about weakly pseudoconvex points. As recently as 1972, experts were still conjecturing that a weakly pseudoconvex point is, up to local biholomorphic equivalence, weakly convex (this would be in analogy with the fact, which in 1972 had been known for nearly twenty years, that a strongly pseudoconvex point is locally biholomorphically equivalent to a strongly convex point). We now realize that such a conjecture is not even approximately correct, at least not in a form that gives useful results. In fact we are only beginning to understand weakly pseudoconvex points.

By the same token, it has long been known that weakly pseudoconvex domains can be exhausted by smoothly bounded strongly pseudoconvex domains. It was generally believed that weakly pseudoconvex domains could be approximated from the outside by strongly pseudoconvex domains. That such is not the case (see [DF1]) came as quite a shock. See Section 1 for further details.

In the next three sections I would like to use items (i) - (v) listed above as a vehicle for discussing what we have learned about pseudoconvexity, and what remains to be done. It should be stressed that I am concentrating on only a portion of the theory of several complex variables--that which derives primarily from the Princeton school. But it is this collection of ideas with which I am most familiar, and which to my mind has yielded the most novel ideas and techniques in the past decade or so. While the inner functions problem, the corona problem, and many other programs have also had strong effects on several complex variables, they are not salient to the theme of this paper and I shall not discuss them.

The reader is advised to refer to [HO] or [KR1] or [RAN2] for basic definitions which bear on the discussion that follows.

Section 1 The Kohn-Nirenberg Example, the Worm Domain, and Related Phenomena

The most important elementary fact about a strongly pseudoconvex point P in the boundary of a domain Ω is the existence of a local

<u>holomorphic</u> <u>separating</u> <u>function</u> for Ω at P. Indeed, if

$$\Omega = \{z \in \mathbb{C}^n : \rho(z) < 0\}$$

satisfies

$$\sum_{j,k=1}^{n} \frac{\partial^2 \rho}{\partial z_j \partial \bar{z}_k}(P) w_j \bar{w}_k \geq C|w|^2 \qquad \forall w \in \mathbb{C}^n$$

then

$$L_P(z) \equiv \sum_{j=1}^{n} \frac{\partial \rho}{\partial z_j}(P)(z_j - P_j) + \sum_{j,k=1}^{n} \frac{\partial^2 \rho}{\partial z_j \partial z_k}(P)(z_j - P_j)(z_k - P_k)$$

satisfies

$$\bar{\Omega} \cap \{z : |z - P| < \epsilon_0 , L_P(z) = 0\} = \{P\}$$

when $\epsilon_0 > 0$ is small. Alternatively, once one notices that there is a local biholomorphic change of coordinates near P which renders $\partial\Omega$ strongly <u>convex</u>, say that in the new coordinates $(w_1,\ldots,w_n)$ the domain Ω near P has defining function $\rho^*(w)$ and that $P \longleftrightarrow P^*$, then it is clear that the pullback of

$$\varphi(w) \equiv \sum \frac{\partial \rho^*(P^*)}{\partial w_j} \cdot (w_j - P^*_j)$$

near P will be a local holomorphic separating function.

The existence of holomorphic separating functions is a critical step in the solution of the Levi problem (see [BER]). In the construction of integral formulas using the Cauchy-Fantappiè machinery, the existence of <u>global</u> holomorphic separating functions (gotten from local holomorphic separating functions by solving a suitable cohomology problem) is fundamental. Holomorphic separating functions provide important information about optimal regularity for the $\bar{\partial}$ problem (see [KR2]). Finally, holomorphic separating functions are very closely related to holomorphic peaking functions which, in turn, are basic for function algebraic considerations.

Were the aforementioned conjecture, that smoothly bounded weakly pseudoconvex domains are locally biholomorphically equivalent to weakly convex domains, in fact true then the pullback of

$$\varphi(z) \equiv \sum \frac{\partial \rho^*(P^*)}{\partial w_j} \cdot (w_j - P^*_j) ,$$

where $\rho^*(w)$ is a defining function for the convex domain, would give a <u>weak</u> local holomorphic separating function h_P at each point of the boundary. This would mean that

$$P \in \bar{\Omega} \cap \{z : |z - P| < \epsilon_0 , h_P(z) = 0\} \subseteq \partial\Omega .$$

In 1973 Kohn and Nirenberg [KON] destroyed this optimistic program by proving that the point (0,0) in the boundary of the smooth, pseudoconvex domain

$$\Omega = \{(z_1,z_2) \in \mathbb{C}^2 : \mathrm{Re}\, z_2 + |z_1 z_2|^2 + |z_1|^8 + \frac{15}{7}|z_1|^2 \mathrm{Re}(z_1)^6 < 0\}$$

has no local holomorphic separating function. Indeed if h is holomorphic in a neighborhood of 0 and $h(0) = 0$ then h vanishes infinitely often <u>on</u> Ω in every neighborhood of 0 . In particular we see that the Kohn-Nirenberg domain is not locally biholomorphically equivalent to a convex domain near 0 . Hakim/Sibony and Sibony (see [HS1],[SI1]) have subsequently obtained stronger examples which show that weakly pseudoconvex boundaries cannot necessarily be made locally convex even with a biholomorphism from one side.

We now understand that the domains constructed by Kohn/Nirenberg and Hakim/Sibony are weakly pseudoconvex domains of the most tractable sort: that is, the the Levi form vanishes at the bad points, but only to finite order. Such points are called finite type (see Section 2) and for many purposes they are as good as strongly pseudoconvex points. The lesson to be learned is that even if one restricts attention to the simplest weakly pseudoconvex domains, even in $\mathbb{C}^2$, the notion that pseudoconvexity is a biholomorphically invariant version of convexity is far too simple-minded.

The Kohn-Nirenberg example inspired a number of people to investigate holomorphic separating functions, peak points, Šilov boundaries, and related phenomena. The papers [HS1], [BL1], [BL2], [SI1], [SI2], [FO1], [BEF] give an overview of some of this work.

Another drive to reduce the study of weakly pseudoconvex domains to the more tractable strongly pseudoconvex domains was the problem of the Nebenhülle. If Ω is a (pseudoconvex) domain then Ω is said to have a <u>Stein neighborhood basis</u> if $\Omega = \cap \Omega_j$, each Ω_j is strongly pseudoconvex, and $\Omega_j \supseteq \Omega_{j+1}$, each j . If Ω does not have a Stein neighborhood basis then

$$\cap \{\Omega' : \Omega' \supseteq \Omega , \Omega' \text{ is strongly pseudoconvex}\}$$

is called the <u>Nebenhülle</u> of Ω .

It was commonly supposed, if not fervently hoped, that every smooth

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pseudoconvex domain has a Stein neighborhood basis. In retrospect, this was probably a bit optimistic. For the Hartogs triangle

$$T = \{(z_1, z_2) : |z_1| < |z_2| < 1\}$$

has a very large Nebenhülle. To be sure, ∂T is only Lipschitz, but there is no substantive reason why smoothly bounded domains should be better behaved.

In any event, it was quite a surprise when in 1977 Diederich and Fornaess [DF1] exhibited the "worm domain": a smoothly bounded pseudoconvex domain with non-trivial Nebenhülle. It was a difficult lesson to accept that pseudoconvex domains look a lot different from the inside than from the outside--in particular they are much more subtle than convex domains. But this remarkable discovery gave a great impetus to the research of the 1970's.

I would be remiss at this point not to mention the one piece of good news that came along in this time period: the discovery of the bounded strictly plurisubharmonic exhaustion functions [DF2]. In fact, given any smoothly bounded pseudoconvex domain Ω then there is a defining function ρ for Ω and an $\eta > 0$ such that $\hat{\rho} \equiv -(-\rho)^{\eta}$ satisfies

(i) $\hat{\rho}$ is strictly plurisubharmonic on Ω ;
(ii) $\hat{\rho} < 0$ on Ω , $\hat{\rho} = 0$ on $\partial\Omega$;
(iii) $\Omega_c \equiv \{z \in \Omega : \hat{\rho} \leqslant c\} \subset\subset \Omega$, all $c < 0$;
(iv) If $K \subset\subset \Omega$ then there is a $c < 0$ such that $K \subseteq \Omega_c$.

For many purposes, the bounded plurisubharmonic exhaustion function is a good substitute for the program that the Kohn-Nirenberg example and the Diederich-Fornaess example killed. It has proved particularly useful in studying the holomorphic mapping problem (see for instance [BEL] and [DF4]).

One very important lesson that was learned from the Kohn-Nirenberg example and the two results of Diederich and Fornaess is that there is no substitute in several complex variables for hard calculations. These papers set the tone for the decade of research that followed.

Section 2 Points of Finite Type and Subelliptic Estimates

Let $\Omega = \{z : \rho(z) < 0\} \subseteq \mathbb{C}^2$ have smooth boundary. A point $P \in \partial\Omega$ is said to be of finite type $m \in \mathbb{Z}^+$ if there is a nonsingular complex variety V such that

$$|\rho(v)| \leqslant C|v - P|^{m+1} \quad , \quad v \in V$$

while there is no nonsingular complex variety V' such that

$$|\rho(v')| \leqslant C|v' - P|^{m+2} \quad , \quad v' \in V' .$$

The notion of finite type is unoriented: it cannot distinguish between pseudoconvexity and pseudoconcavity. Thus it turns out that the only points of type 1 are strongly pseudoconvex points and strongly pseudoconcave points. Pseudoconvex points are always of odd type. In the domain

$$\{(z_1,z_2) : |z_1|^2 + |z_2|^{2k} < 1\} ,$$

boundary points of the form $(e^{i\theta},0)$ are of type $2k - 1$.

The notion of finite type helps us to quantify the idea that strongly pseudoconvex points are generic in the boundaries of smooth pseudoconvex domains. For if $U \subseteq \partial\Omega \subseteq \mathbb{C}^2$ is a relatively open subset containing only points of finite type exceeding one, then U consists only of points where the Levi form vanishes. In other words, U consists only of points where the Levi form has zero rank. It follows (see [KR1]) that U is foliated by one dimensional complex manifolds. As a result, each point of U is of infinite type, and that is a contradiction.

Continuing to restrict attention to $\mathbb{C}^2$, we now give another (equivalent) definition of finite type. If $\Omega = \{z: \rho(z) < 0\}$ is a smoothly bounded domain in $\mathbb{C}^2$, $P \in \partial\Omega$, and $\frac{\partial\rho}{\partial z_2}(P) \neq 0$, we define a vector field in a neighborhood of P by

$$L = \frac{\partial\rho}{\partial z_2}(P)\frac{\partial}{\partial z_1} - \frac{\partial\rho}{\partial z_1}(P)\frac{\partial}{\partial z_2} .$$

Then L, $\bar{L}$ span (over $\mathbb{R}$) the <u>complex tangent space</u> to $\partial\Omega$ at points near P. Their span has no component in the <u>complex normal</u> direction

$$Z = \mathrm{Im}\left[\frac{\partial\rho}{\partial z_1}(P)\frac{\partial}{\partial z_1} + \frac{\partial\rho}{\partial z_2}(P)\frac{\partial}{\partial z_2}\right] .$$

However define

$$\mathcal{L}_0 = \mathrm{span}_{\mathbb{R}}\{L,\bar{L}\}$$
$$\mathcal{L}_1 = \mathrm{span}_{\mathbb{R}}\{\mathcal{L}_0,[\mathcal{L}_0,L],[\mathcal{L}_0,\bar{L}]\}$$

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$$\mathcal{L}_j = \text{span}_{\mathbb{R}}\{\mathcal{L}_{j-1}, [\mathcal{L}_{j-1}, L], [\mathcal{L}_{j-1}, \bar{L}]\} .$$

We call P a point of finite type m if $\mathcal{L}_{m-1}$ contains no element with non-zero component in the direction Z while $\mathcal{L}_m$ does contain such an element.

Implicit in Kohn's paper [KO1] is the fact that the two definitions of finite type which we have given, one in terms of order of contact of non-singular varieties and the other in terms of commutators of vector fields, are equivalent. The main thrust of Kohn's paper [KO1] was to show that, in $\mathbb{C}^2$, finite type points P are precisely those near which a subelliptic estimate for the $\bar{\partial}$-Neumann problem of the form

$$|||u|||_\epsilon^2 \leqslant C[\|\bar{\partial}u\|^2 + \|\bar{\partial}^* u\|^2]$$

holds. Here u is a test function supported in a neighborhood of P, $|||u|||_\epsilon$ is a tangential Sobolev norm of order ϵ, and $\| \ \|$ is the 0-order Sobolev (or L^2) norm. Kohn estimated ϵ in terms of the type m and subsequent work in [GR] and [KR2] showed that this estimate is sharp.

Since 1972, there has been a great deal of work to determine the correct analogue of "finite type", to determine the sharp value for ϵ, and also to determine the right necessary and sufficient conditions for subelliptic estimates for the $\bar{\partial}$-Neumann problem, in dimensions exceeding two. Bloom and Graham [BG] formulated a definition of finite type (in any dimension) in terms of order of contact of complex hypersurfaces and proved this definition to be equivalent to one in terms of commutators of vector fields. While this notion of type was helpful to those thinking about peak points and holomorphic support functions (see [BL1],[BL2],[HS1]), it soon became clear that it was <u>not</u> the right condition for subelliptic estimates.

Kohn's important work in [KO2] gave a sufficient condition, in terms of ideals of forms, for the existence of subelliptic estimates for the $\bar{\partial}$-Neumann problem on forms in any dimension. He conjectured that his condition would also be necessary. At about the same time, D'Angelo and Catlin initiated a deep and protracted study of the program which Kohn initiated.

In a series of papers, D'Angelo developed from first principles an algebro-geometric theory of points of finite type. His semi-continuity result in [DAN] signalled that he had found the right theoretical framework. Meanwhile, Catlin built on D'Angelo's ideas and use his own deep insights into the construction of plurisubharmonic functions as a tool for attacking the problem of subelliptic estimates. From the work of D'Angelo and Catlin there evolved the following definition of finite type in $\mathbb{C}^n$:

__Definition:__ Let $\Omega = \{z \in \mathbb{C}^n : \rho(z) < 0\}$ and $P \in \partial\Omega$. If $\gamma: \mathbb{C} \longrightarrow \mathbb{C}^n$ is holomorphic and $\gamma(0) = P$ then define

$$\tau(\gamma) = \frac{\nu(\rho \circ \gamma)}{\nu(\gamma)} ,$$

where $\nu(*)$ denotes the order of vanishing of $*$. Define the __type__ $T(P)$ of P to be

$$T(P) \equiv \sup_{\gamma} \tau(\gamma) .$$

We say that P is of __finite type__ if and only if $T(P) < \infty$.

Catlin [CAT2] has proved that if $\partial\Omega$ is pseudoconvex near P then a subelliptic estimate holds near P for the $\bar{\partial}$-Neumann problem if and only if P is of finite type. Diederich and Fornaess [DF3] have proved that a bounded pseudoconvex domain with real analytic boundary is of finite type. This means that, given a bounded domain Ω with real analytic boundary, there is a number $M > 0$ such that each point of $\partial\Omega$ has finite type not exceeding M . These results represent some of the most important progress made in several complex variables in the last fifteen years. They provide us with a large collection of domains on which the basic constructions of complex function theory can (at least in principle) be performed. Section 3 contains some particularly dramatic applications of these results.

It is my view that an important direction of future research ought to be the detailed study of harmonic analysis on domains of finite type. The development of Fatou theorems, Lusin area integrals, and admissible maximal functions in the special case of strongly pseudoconvex domains has already led to new understanding of singular integral operators, covering theorems, spaces of homogeneous type, and the other tools of harmonic analysis (see [ST1], [FOS1]). More recently, progress has been made on domains of finite type in $\mathbb{C}^2$ (see [NSW1],[KR4]). It is becoming increasingly clear that invariant metrics, such as the Bergman, Caratheodory, and Kobayashi metrics, will play a vital role in the final understanding of these issues ([KR3],[KR4]). A careful understanding of Catlin's work should lead to new developments on domains of finite type in $\mathbb{C}^n$, $n > 2$.

The papers of D'Angelo in this volume provide a valuable discussion of points of finite type, both from the point of view of commutators and the point of view of algebraic geometry. I hope that they will enable a new generation of researchers to consider (i) the behavior of invariant metrics near boundary points of finite type (see [CAT1] for results in $\mathbb{C}^2$), (ii) the theory of quadratic integrals, such as the Littlewood-Paley g function and the Lusin area integral, near ponts of finite type (see [ST1], [NSW1], [KR4]), (iii) boundary behavior of holomorphic functions near points of finite type (see [NSW1],[KR3], [KR4]), (iv) the theory of "real Hardy spaces" near points of finite type (see [FES], [FOS2]), (v) asymptotic expansions for canonical kernels (such as the Bergman, Szegö, and Neumann kernels) near poins of finite type (see [GRS], [NSW3]), and (vi) construction of non-canonical kernels (such as the Henkin-Ramirez kernel) near points of finite type (see [GRS], [FO2] for results in $\mathbb{C}^2$).

Section 3 The Fefferman Mapping Theorem and Related Results

In [FE1], C. Fefferman proved the following striking result:

Theorem: Let Ω_1 and Ω_2 be smoothly bounded, strictly pseudoconvex domains in $\mathbb{C}^n$. Let $\Phi: \Omega_1 \to \Omega_2$ be a biholomorphic map. Then Φ extends to a diffeomorphism of $\bar{\Omega}_1$ to $\bar{\Omega}_2$.

Fefferman's proof involves a detailed analysis of the asymptotic behavior of Bergman metric geodesics at the boundary and the details are too cumbersome to be presented here. More relevant for our purposes is a consideration of the impact of this theorem. Besides the solution of the Levi problem (and related topics such as the $\bar{\partial}$-Neumann problem), this was one of the very first theorems proved about a class of domains. It was certainly the first such result about holomorphic mappings. Prior to Fefferman, consideration of holomorphic mappings proceeded by explicit calculation of mappings of explicitly given domains described by polynomial inequalities. While these calculations were often quite difficult (see [HUA]) and involved powerful machinery (e.g. the Lie theory in the program of Cartan--see [HEL]), they represent what can now be safely called a classical chapter in complex analysis. Even though the details of Fefferman's proof are extremely difficult, it should be stressed that the statement of the theorem is readily accessible; moreover the very form of Fefferman's results already leads to new insights.

As an instance of this last remark, Klembeck [KL] used Fefferman's asymptotic formula for the Bergman kernel to calculate the asymptotic boundary behavior of the Bergman metric on a strongly pseudoconvex domain. One result is that the holomorphic sectional curvature of the metric approaches that of the ball. Coupled with a result of Lu Qi-Keng about complex manifolds of constant holomorphic sectional curvature, this yields a new proof of Bun Wong's theorem: the only strongly pseudoconvex domain with transitive automorphism group is the ball. Greene and Krantz ([GK1], [GK2], [GK3]), using Kelmbeck's work as inspiration, were able to do a more detailed analysis of Fefferman's asymptotic expansion and thus to learn how the group of biholomorphic maps of a domain depends on the boundary of that domain.

Probably the most profound consequence of Fefferman's theorem is that it vindicates a program of Poincaré to calculate (biholomorphic) differential boundary invariants for strongly pseudoconvex domains. A formal argument with power series ([FE1],[GK4]) shows that the invariants exist in principle, but one needs to know that biholomorphisms extend smoothly to the boundary before these "potential invariants" can be considered true invariants. Immediately following Fefferman's result, Chern and Moser [CM] completed Poincaré's program in principle in that they showed how the invariants can be calculated. Their results were to some extent anticipated by those of Tanaka in [TAN]. Burns, Shnider, and Wells ([BSW],[BS]) used the invariants to obtain important insights into the biholomorphic self-maps of domains. Perhaps the deepest work on the boundary invariants has been done by Fefferman himself in [FE2], [FE3]. In these papers Fefferman gave, among other things, an effective procedure for deciding whether two strongly pseudoconvex boundary points are biholomorphically equivalent. It should be stressed that Fefferman's results here, while very complete, apply only to biholomorphic equivalence <u>at a single point</u>. Much less is known about global obstructions to biholomorphic equivalence (however see [GK2], [GK3], [GK4],[BED1],[BDA]). For weakly pseudoconvex domains, even though there are many results about smoothness to the boundary of biholomorphic mappings, essentially nothing is known about biholomorphic differential invariants in the boundary. Even the case of finite type 3 in $\mathbb{C}^2$ has not been developed. Clearly there is much important work to be done in this area.

Fefferman's theorem has also inspired many people to consider both extending and simplifying the biholomorphic mapping theorem. A very detailed survey of this work was given by Bedford in [BED2], and I shall only make a few remarks about it here.

The explicit nature of Fefferman's asymptotic expansion required the creation of a delicate calculus of singular integrals (Boutet de

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