Lecture Notes in Mathematics 1659

Vladimir Kozlov Vladimir Maz'ya

Theory of a Higher-Order Sturm-Liouville Equation

Springer

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Authors

Vladimir Kozlov
Department of Mathematics
University of Linköping
58183 Linköping
Sweden
e-mail: vlkoz@math.liu.se

Vladimir Maz'ya
Department of Mathematics
University of Linköping
58183 Linköping
Sweden
e-mail: vlmaz@math.liu.se

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Introduction

The present book is devoted to the ordinary differential equation

$$(\partial_t + k_+)^{m_+}(-\partial_t - k_-)^{m_-} w(t) - \omega(t)w(t) = f(t), \tag{0.1}$$

where ∂_t is the differentiation with respect to the variable t, and k_+, k_- are real numbers, $k_+ > k_-$, and m_+, m_- are positive integers. In the case $k_+ = -k_- = k$ and $m_+ = m_- = 1$ the equation (0.1) becomes the classical Sturm-Liouville equation

$$-w''(t) + (k^2 - \omega(t))w(t) = f(t),$$

which is of importance both in mathematics and in physics. We shall assume a priori that the coefficient ω in (0.1) is measurable and non-negative, and introduce various assumptions about ω in different parts of the book.

Whereas asymptotic properties of solutions to the Sturm-Liouville equation as well as their oscillation and positivity properties have been intensively studied, for the time being there are no similar results for the equation (0.1).

The purpose of the present book is to develop a detailed theory of the equation (0.1), which includes

- conditions of solvability
- classes of uniqueness
- positivity properties of solutions and Green's functions
- asymptotic properties of solutions at infinity.

Being of independent interest, the equation (0.1) also proved to have important applications to differential equations with operator coefficients and elliptic boundary value problems for domains with non-smooth boundaries. To be more specific, solutions of (0.1) serve as majorants for solutions of operator equations and boundary value problems.

The first chapter is devoted to the equation

$$(\partial_t + k_+)^{m_+}(-\partial_t - k_-)^{m_-} w(t) = f(t) \tag{0.2}$$

on $\mathbb{R}$. The starting point is a construction of Green's function, which appears to be positive. From two-sided estimates for Green's function we obtain sufficient conditions for the solvability of (0.2) both necessary and sufficient in the case $f \geq 0$.

In Chapter 2 we consider the equation (0.2) on the positive semi-axis complemented by the boundary conditions

$$(\partial_t + k_+)^j w|_{t=0} = f_j,\ j = 0\ldots, m_+ - 1.$$

We obtain the following "positivity principle" for arbitrary solutions of the non-homogeneous problem. If $f \geq 0$ on $(0, \infty)$ and $f_j \geq 0$ then $w > 0$ on $(0, \infty)$. A similar positivity principle is proved for the equation (0.2) on the bounded interval. (In the case of the second-order equation (0.2) such results are, of course, obvious corollaries of the maximum principle).

In Chapters 3-7 we consider various properties of the equation (0.1) on $\mathbb{R}$ with non-zero ω. We start with the case ω = const (Chapter 3), then consider variable $\omega \geq 0$ subject to a certain mild condition formulated in terms of Green's function for the equation (0.2) (Chapters 4 and 5). We find Green's function for the perturbed equation (0.1) and prove very general existence and uniqueness results, formulated in terms of this Green's function. We construct solutions under very weak assumptions about the function f and prove that they belong to a uniqueness class. In Chapter 5 we also extend the positivity principle to the equation (0.1) on $\mathbb{R}_+$.

Abstract formulations derived in Chapters 4 and 5 are made more visual in Chapter 6, where we consider several special classes of ω. In Chapter 7 we find asymptotic representations of solutions to the equation (0.1) at infinity. We show, for example, that the solution w of the homogeneous equation (0.1) satisfies the asymptotic formula

$$w(t) \sim \text{const } e^{-k_\pm t} t^\sigma,$$

where $0 \leq \sigma \leq m_\pm - 1$, if and only if

$$\int_1^\infty \tau^{m_\pm - 1} \omega(\tau) d\tau < \infty$$

It seems that the equation (0.1) did not attract special attention before. However there exists rich bibliography concerning qualitative and asymptotic theories for higher order linear ordinary differential equations with variable coefficients. We mention only the books by Kiguradze, Chanturia (1993), Eastham (1989), Lomov (1992) and papers Levin (1969) and Eloe, Ridenhour (1994) where readers can find more references.

In Appendix we briefly discuss one of the mentioned already applications of previous results. We consider a class of ordinary differential equations

$$\sum_{k=0}^{k=l} A_k(t) \frac{d^k}{dt^k} u(t) = F(t) \tag{0.3}$$

with operator coefficients acting in pairs of Hilbert spaces first studied by Agmon, Nirenberg (1963). It appears that solutions of (0.3) satisfy the following "comparison principle"

$$||u||_{W^l(t,t+1)} \leq \text{const}\ w(t) \tag{0.4}$$

where W^l is a certain abstract Sobolev space on the interval $(t, t+1)$. The majorant $w(t)$ in (0.4) satisfies the equation (0.1) with

$$f(t) = ||F||_{L_2(t,t+1;H_0)}$$

where the norm is taken in the space of abstract functions with values in a Hilbert space H_0.

Hence the information on the equation (0.1) obtained in the book leads to "pointwise" estimates for solutions of the operator differential equation (0.3).

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Table of Contents

1. Basic Equation with Constant Coefficients

1.1 Introduction

Let k_-, k_+ be real numbers and m_+, m_- be positive integers. In this chapter we shall study the equation

$$(\partial_t + k_+)^{m_+}(-\partial_t - k_-)^{m_-} w(t) = f(t) \quad \text{on} \quad \mathbb{R}. \tag{1.1}$$

In Sections 1.2 and 1.3 we study Green's function for (1.1) and show that the condition

$$\int_{R^1_\pm} e^{k_\mp \tau}(1+ \mid \tau \mid)^{m_\mp - 1} \mid f(\tau) \mid dt < \infty \tag{1.2}$$

is sufficient and in the case $f \geq 0$ necessary for the existence of the solution w to (1.1) which satisfies

$$\lim_{t \to \pm\infty} e^{k_\mp t} \mid w(t) \mid = 0$$

(Proposition 1.3.1). The less restrictive condition

$$\liminf_{t \to \pm\infty} e^{k_\mp t} \mid w(t) \mid = 0 \tag{1.3}$$

describes a uniqueness class of solutions to (1.1) with f subject to (1.2). In Proposition 1.4.1 we prove that the requirement

$$\int_0^\infty e^{k_+ \tau}(1 + \tau)^j \mid f(\tau) \mid d\tau < \infty \tag{1.4}$$

for some $j = 0, \dots, m_+ - 1$, along with (1.2) implies the asymptotic representation

$$w(t) = e^{-k_+ t}\Big(\sum_{s=m_+ - 1 - j}^{m_+ - 1} a_s t^s + O\big(t^{m_+ - 1 - j}\big)\Big) \quad \text{as} \quad t \to +\infty.$$

Moreover, (1.4) is necessary for this representation if $f \geq 0$.

1.2 Green's Function for $\mathcal{M}(\partial_t)$ on $\mathbb{R}$

Let k_+, k_- be real numbers, $k_+ > k_-$, and let m_+, m_- be positive integers. We denote

$$\mathcal{M}(\partial_t) = (\partial_t + k_+)^{m_+}(-\partial_t - k_-)^{m_-}.$$

All solutions of the homogeneous equation

$$\mathcal{M}(\partial_t)\xi(t) = 0, \quad t \in \mathbb{R},$$

are

$$\xi(t) = e^{-k_+t}\Pi_+(t) + e^{-k_-t}\Pi_-(t), \tag{1.5}$$

where $\Pi_\pm$ are arbitrary polynomials of degrees $m_\pm - 1$.

We denote by g Green's function for the operator $\mathcal{M}(\partial_t)$, i.e. the solution of the equation

$$\mathcal{M}(\partial_t)g(t) = \delta(t) \quad \text{on} \quad \mathbb{R}, \tag{1.6}$$

subject to

$$g(t) = o\big(e^{-k_\mp t}\big) \quad \text{as} \quad t \to \pm\infty,$$

where δ is Dirac's function.

Lemma 1.2.1. *The following formulae hold:*

$$g(t) = e^{-k_+t}\sum_{q=0}^{m_+-1}\frac{t^q}{q!}\binom{m_+ + m_- - 2 - q}{m_- - 1}(k_+ - k_-)^{-m_+-m_-+1+q} \tag{1.7}$$

for $t \geq 0$ *and*

$$g(t) = e^{-k_-t}\sum_{q=0}^{m_--1}\frac{(-t)^q}{q!}\binom{m_+ + m_- - 2 - q}{m_+ - 1}(k_+ - k_-)^{-m_+-m_-+1+q} \tag{1.8}$$

for $t < 0$.

Proof. By using the Fourier transform we obtain

$$g(t) = i^{m_- - m_+}\frac{1}{2\pi}\int_{\Im\lambda=\beta}\frac{e^{i\lambda t}d\lambda}{(\lambda - ik_+)^{m_+}(\lambda - ik_-)^{m_-}}, \tag{1.9}$$

where $\beta \in (k_-, k_+)$. For $t > 0$ we have

$$\begin{aligned} g(t) &= ie^{-k_+t}i^{m_--m_+}\sum_{s=0}^{m_+-1}\frac{(it)^{m_+-1-s}}{(m_+ - 1 - s)!} \\ &\times \frac{(-m_-)(-m_- - 1)\dots(-m_- - s + 1)}{s!}(ik_+ - ik_-)^{-m_--s} \end{aligned}$$

which yields (1.7).

If $t < 0$ then

$$\begin{aligned} g(t) &= -ie^{-k_- t} i^{m_- - m_+} \sum_{s=0}^{m_- -1} \frac{(it)^{m_- -1-s}}{(m_- - 1 - s)!} \\ &\times \frac{(-m_+)(-m_+ - 1)\dots(-m_+ - s + 1)}{s!} (ik_- - ik_+)^{-m_+ - s} \end{aligned}$$

and we arrive at (1.8). □

We collect several simple properties of $g(t)$ which follow directly from (1.7), (1.8) and (1.6).

Proposition 1.2.2. (i) *The following inequalities hold:*

$$c_1 e^{-k_+ t}(1+t)^{m_+ -1} \leq g(t) \leq c_2 e^{-k_+ t}(1+t)^{m_+ -1}$$

for $t \geq 0$,

$$c_1 e^{-k_- t}(1+|t|)^{m_- -1} \leq g(t) \leq c_2 e^{-k_- t}(1+|t|)^{m_- -1}$$

for $t \leq 0$.

(ii) *For all* $t \in \mathbb{R}$ *and* $j = 0, 1, \dots, m_+ + m_- - 1$

$$\left|\partial_t^j g(t)\right| \leq c\, g(t). \tag{1.10}$$

(iii) *The inequality*

$$g(t_1) \leq c\, e^{-k_\mp (t_1 - t_2)} g(t_2)$$

holds for $t_1 \gtrless t_2$.

(iv) *For all* $t, \tau \in \mathbb{R}$, $t\tau \leq 0$

$$g(t-\tau) \leq c\, g(t) g(-\tau).$$

(v) *There exist constants* c_1, c_2, *such that*

$$c_1 e^{-k_- t} \leq \sup_{\tau \in \mathbb{R}} \frac{g(t-\tau)}{g(-\tau)} \leq c_2 e^{-k_- t} \quad \text{for} \quad t \geq 0,$$

$$c_1 e^{-k_+ t} \leq \sup_{\tau \in \mathbb{R}} \frac{g(t-\tau)}{g(-\tau)} \leq c_2 e^{-k_+ t} \quad \text{for} \quad t \leq 0.$$

The following assertion shows that one can apply some differential operators to g with preservation of positivity.

Proposition 1.2.3. *The inequality holds*

$$(\partial_t + k_+)^n (-\partial_t - k_-)^j g(t) > 0 \quad \text{for} \quad n < m_+,\ j < m_-. \tag{1.11}$$

Moreover,

$$(\partial_t + k_+)^{m_+} \left(- \partial_t - k_-\right)^j g(t) \quad \text{is positive for} \quad t < 0$$

and equals 0 *for* $t \geq 0$ *if* $j < m_-$. *Besides* $(\partial_t + k_+)^n (-\partial_t - k_-)^{m_-} g(t)$ *equals* 0 *for* $t \leq 0$ *and is positive for* $t > 0$ *if* $n < m_+$.

Proof. Introduce the polynomial

$$P(k,n;\tau) = \sum_{q=0}^{n-1} \frac{\tau^q}{q!} \binom{k+n-2-q}{k-1} \tag{1.12}$$

for $k \geq 1,\ n \geq 1$, and

$$P(0,n;\tau) = \frac{t^{n-1}}{(n-1)!},\ P(k,0;\tau) = 0$$

for $n \geq 1,\ k \geq 1$. It can be verified directly, that

$$\big(-\partial_\tau + 1\big) P(k,n;\tau) = P(k-1,n;\tau) \tag{1.13}$$

and

$$\partial_\tau P(k,n;\tau) = P(k,n-1;\tau). \tag{1.14}$$

Since

$$g(t) = (k_+ - k_-)^{1-m_+-m_-} e^{-k_+ t} P(m_-, m_+; (k_+ - k_-)t) \tag{1.15}$$

for $t \geq 0$ and

$$g(t) = (k_+ - k_-)^{1-m_+-m_-} e^{-k_- t} P(m_+, m_-; (k_- - k_+)t) \tag{1.16}$$

for $t \leq 0$, we have for $t > 0$

$$(\partial_t + k_+)^n (-\partial_t - k_-)^j g(t) = (k_+ - k_-)^{n+j+1-m_+-m_-} e^{-k_+ t} \times P(m_- - j, m_+ - n; (k_+ - k_-)t) \tag{1.17}$$

and for $t < 0$

$$(\partial_t + k_+)^n (-\partial_t - k_-)^j g(t) = (k_+ - k_-)^{n+j+1-m_+-m_-} e^{-k_- t} \times P(m_+ - n, m_- - j; (k_- - k_+)t). \tag{1.18}$$

□

1.3 Necessary and Sufficient Condition for Solvability

Consider the equation

$$\mathcal{M}(\partial_t) w = f \quad \text{on} \quad \mathbb{R}, \tag{1.19}$$

where $f \in L_{1,\mathrm{loc}}(\mathbb{R})$ and w belongs to the class $W_{1,\mathrm{loc}}^{m_+ + m_-}(\mathbb{R})$, i.e. it is absolutely continuous with its derivatives up to order $m_+ + m_- - 1$.

We prove an existence result.

Proposition 1.3.1. *The condition*

$$\int_{-\infty}^{\infty} g(-\tau) \mid f(\tau) \mid d\tau < \infty$$

or, which is the same,

$$\int_0^{\infty} e^{k_-\tau}(1+\tau)^{m_- -1}|f(\tau)|d\tau < \infty, \tag{1.20}$$

and

$$\int_{-\infty}^{0} e^{k_+\tau}(1+|\tau|)^{m_+ -1}|f(\tau)|d\tau < \infty \tag{1.21}$$

is sufficient and in the case $f \geq 0$ necessary for the existence of the solution w of (1.19) *subject to* (1.3). *This solution is represented in the form*

$$w(t) = \int_{-\infty}^{\infty} g(t-\tau)f(\tau)d\tau \tag{1.22}$$

and satisfies

$$\partial_t^j w(t) = \begin{cases} o(e^{-k_- t}) & as \quad t \to +\infty, \\ o(e^{-k_+ t}) & as \quad t \to -\infty, \end{cases} \tag{1.23}$$

where $j = 0, 1, \ldots, m_+ + m_- - 1$.

Proof. Let (1.20) and (1.21) be fulfilled and w be given by (1.22). Then by Proposition 1.2.2 (ii), (iii)

$$\left|\partial_t^j w(t)\right| \leq c \max\left\{e^{-k_- t},\, e^{-k_+ t}\right\} \int_{\mathbb{R}} g(-\tau)|f(\tau)|d\tau$$

for $j = 0, 1, \ldots, m_+ + m_- - 1$. Hence w has $m_+ + m_- - 1$ locally bounded derivatives.

Since

$$\left(\partial_t + k_+\right)^{m_+ -1}\left(-\partial_t - k_-\right)^{m_-} w(t) = \int_{-\infty}^{t} e^{-k_+(t-\tau)} f(\tau)d\tau,$$

the $\left(m_+ + m_- - 1\right)$-th derivative of w is absolutely continuous and w satisfies (1.19).

Let us prove (1.23). By Proposition 1.2.2 (ii) and (i), we have

$$\mid \partial_t^j w(t) \mid \leq c\Big(\int_{-\infty}^{t} e^{-k_+(t-\tau)}(1+t-\tau)^{m_+ -1}|f(\tau)|d\tau + \int_t^{\infty} e^{-k_-(t-\tau)}(1+\tau-t)^{m_- -1}|f(\tau)|d\tau\Big) \tag{1.24}$$

for large positive t. The first integral in (1.24) can be estimated by