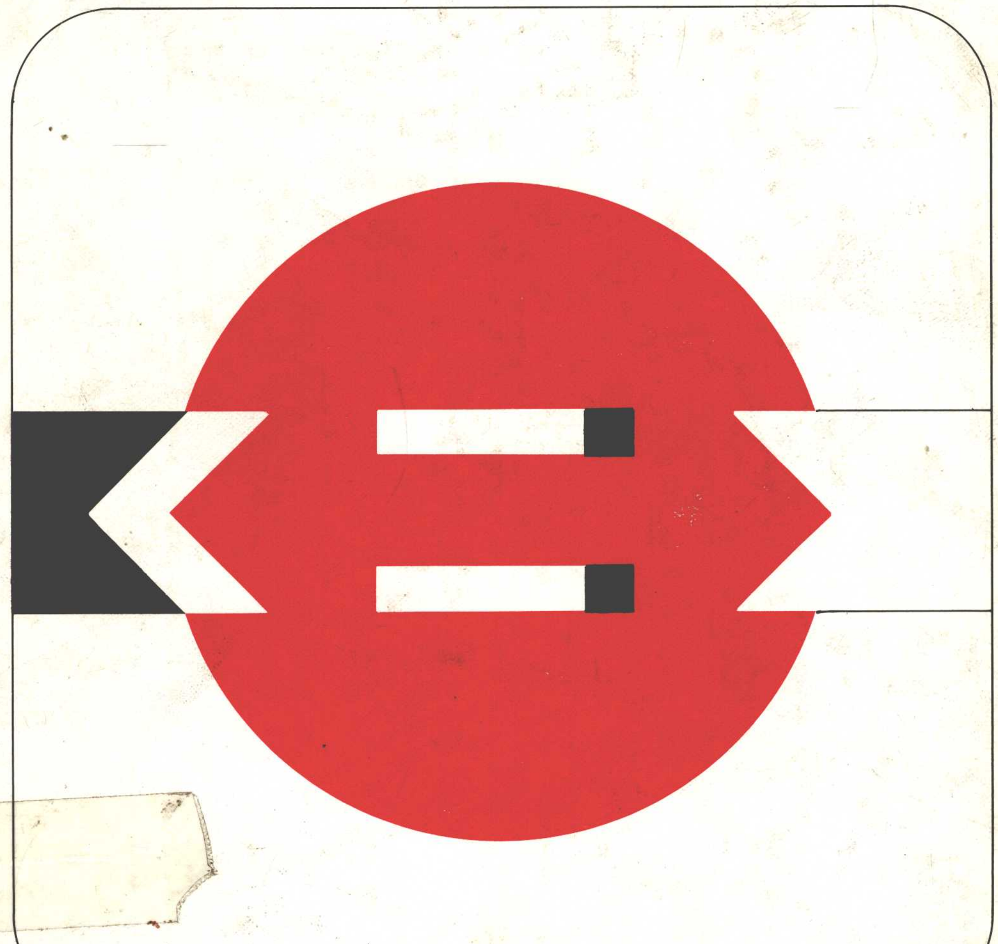
Elementary Linear Algebra with Applications



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Elementary Linear Algebra with Applications

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To my wife, Maxine; and to my children, Kevin, Todd, and Pam

Preface

This book is written for students in a first course in linear algebra at the sophomore level and is intended primarily for mathematics majors, engineers, science students, and business and economics majors.

The question may properly be asked: Why another textbook in linear algebra? I believe that many of the present books in this subject are too abstract, others are computationally oriented to the point that the mathematics is ignored, and many, if not most, have omitted the *applications* of linear algebra. I have written this book with the idea of achieving a balance among computational skills, applications, and the theory of linear algebra. At the same time I have tried to keep the reading level of the text at a sophomore or even a freshman level.

Chapters 1 and 2 are geometric in nature. The discussion begins with line vectors. The operations on line vectors are used to motivate the definitions of operations on 2- and 3-tuples and these definitions are extended to n-tuples. Proofs in the first two chapters are less rigorous and are geometrically oriented. Much of the material in the first two chapters may have already been covered in a calculus and analytic geometry course. I have started with this material because I believe it is the least abstract and provides the most concrete path to the ideas of linear algebra.

Chapter 3 discusses reduction methods for solving systems of linear equations and also the algebra of matrices. Numerous applications are integrated with the material. Chapter 4 is a short chapter on determinants. I believe that the method I have chosen for defining determinants, which avoids permutations, leads more easily to the theory and also to the techniques for calculating determinants.

Chapter 5 deals with the theory of abstract (real) vector spaces. The reader has already seen numerous examples of vector spaces in the first three chapters. Various function spaces are introduced so that the reader will realize that there are important vector spaces other than n-tuples. Proofs of most of the theorems are given but are often illustrated first with an example. This gives the instructor the option of discussing the example or the proof of the theorem, which largely imitates the example.

Chapter 6 discusses linear transformations and their relationship to matrices. Chapter 7 discusses eigenvalues and eigenvectors and the diagonalization process. Chapter 8 discusses the general inner product. Applications are included in Chapters 6, 7, and 8.

I have tried to motivate concepts through examples, applications, and other means. For instance, I show that the definition of matrix multiplication arises in a natural way as the result of a certain substitution process involving systems of linear equations. This is what led Arthur Cayley to the definition of matrix multiplication in 1858.

Over 165 examples are given in the text. These examples include applications and methods for solving problems. Answers to all the odd-numbered computational exercises are given at the end of the text. The answers to the even-numbered problems are available to instructors in a separate answer book.

It is assumed that the reader has a knowledge of precalculus mathematics including the idea of a function. (A brief review of functions is offered in Section 6.1.) With the exception of some examples and problems (that can be omitted) and the final section of the text, which requires a knowledge of integration techniques, a knowledge of elementary calculus *is not needed.*

This book is an outgrowth of lecture notes that were used during the years 1974–1977 at the University of Wisconsin–Superior. I have taught all the material at a leisurely pace in five semester hours and believe that the entire text can be covered in four semester hours. By a judicious selection of topics, most of the book can be covered in three semester hours. Excluding Sections 4.3, 6.8, 7.5, and 8.3 (which are optional) there are 39 sections. Most of these sections can be covered in a single class period. If a shorter course is desired, Sections 2.4 and 2.5 could be omitted as well as some of the applications. Various options are possible, some of which follow:

1. Start with Chapter 1 and continue through the text. This would work well if the text is used concurrently with or prior to a third semester of calculus.
2. If the text is used following a third semester of calculus, the instructor may wish to cover the first two chapters rapidly or omit Chapters 1 and 2 altogether (allowing students to review this material on their own) and start with Chapter 3.

3. Start with Chapter 3 and continue through Section 3.3; then pick up Chapter 1 and Sections 2.1–2.2, leaving the remainder of Chapter 2 until just before Chapter 8.

I gratefully acknowledge the helpful comments and suggestions made by the reviewers: Professor Jan Jaworowski of Indiana University, Professor Jack Goldberg of the University of Michigan, and Professor Robert Weber of Yale University. I am especially appreciative for the many suggestions and the encouragement provided by Professor David E. Kullman of Miami University, Oxford, Ohio, and for the constructive criticism and perceptive comments of Professor Robert E. Mosher, formerly of California State University at Long Beach. Both Professors Kullman and Mosher carefully read the entire manuscript and are responsible for many improvements in the text. I wish also to thank my friend and colleague, Professor Robert E. Dahlin at the University of Wisconsin-Superior, who provided helpful comments and with whom I have had many mathematical discussions.

For typing the first draft I wish to express appreciation to Grace Collins and Hondoko Tingsantoso. I also wish to thank John J. McCanna, regional editor for Prentice-Hall, who gave me encouragement when I first started this project. Finally, I express appreciation to my mathematics editor, Harry H. Gaines; my production editor, Eleanor Henshaw Hiatt; and the production staff at Prentice-Hall.

FRANCIS G. FLOREY

Contents

1 Line Vectors and Coordinate Vectors

Linear algebra is, in a sense, a study of vectors. From a mathematical point of view, vectors come in many different forms, but all must share certain common properties. Probably the first idea of vector that most of us are exposed to is the notion that a vector is a quantity that has a length and a direction. It is with this informal idea that we begin. As we progress further through the text, we shall give a precise mathematical definition of vector. It will become clear that our initial introduction to vector as a quantity having length and direction is only one example of a much broader classification of objects that we shall call vectors.

1.1 VECTOR ADDITION AND SCALAR MULTIPLICATION

In science, physical quantities such as force, velocity, displacement (movement of a particle from one point to another), and acceleration are described by a magnitude and a direction. The term *vector* is used to identify such a quantity.

Geometrically, a vector can be represented by a directed line segment or arrow. The length of the line segment denotes the magnitude of the vector; the direction of the arrow denotes the direction of the vector. For example, a force of 8 lb could be represented by an arrow 8 units long in the direction of the force (Figure 1.1).

One can draw many arrows 8 units long and in the same direction as the force. All such directed line segments represent the same vector. As

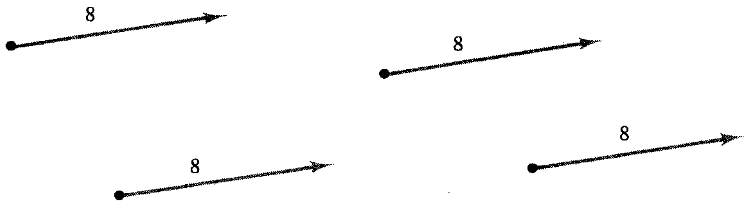


Figure 1.1 An 8-pound force

geometric entities they are different sets of points, but as vector representations they are equal.

Definition 1.1. Two **directed line segments** (arrows) of nonzero length **represent the same vector** if and only if they have the same length and the same direction. The term **line vector** will also be used to refer to the vector represented by a directed line segment.

In the text we shall denote a line vector by a letter with a bar over it, for example, $\bar{v}$ denotes the "vector v" (see Figure 1.2). If the *tail* and *head* of $\bar{v}$ are points A and B, respectively, we shall also write $\bar{v} = \overrightarrow{AB}$.

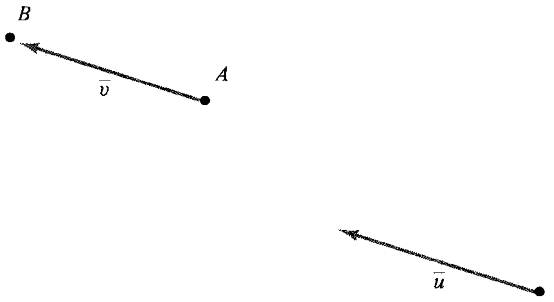


Figure 1.2 Equality of vectors $\bar{v} = \overrightarrow{AB} = \bar{u}$

Addition of Vectors

There are two equivalent procedures that can be used for adding vectors. As Figure 1.3 suggests, from the head of $\bar{u}$ draw $\bar{v}$. The vector $\bar{u} + \bar{v}$ is the vector from the tail of $\bar{u}$ to the head of $\bar{v}$. This method of vector addition is called the **triangle rule of addition.**

An alternative but equivalent procedure is the parallelogram rule for addition (Figure 1.4). Using the **parallelogram rule** to obtain $\bar{u} + \bar{v}$, we draw representations for $\bar{u}$ and $\bar{v}$ from the same point (the tails of $\bar{u}$ and $\bar{v}$ coincide), and then complete the parallelogram. The diagonal drawn from the common point represents $\bar{u} + \bar{v}$.

It is clear from Figure 1.4 and the triangle rule that vector addition is **commutative**, that is,

$$\bar{u} + \bar{v} = \bar{v} + \bar{u}$$

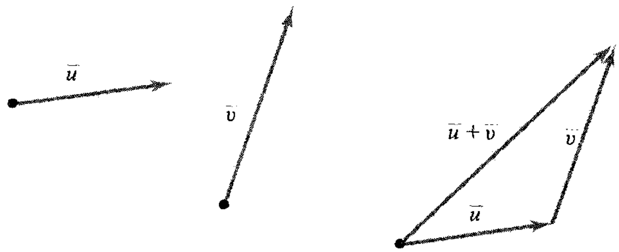


Figure 1.3 Addition of vectors (triangle rule)

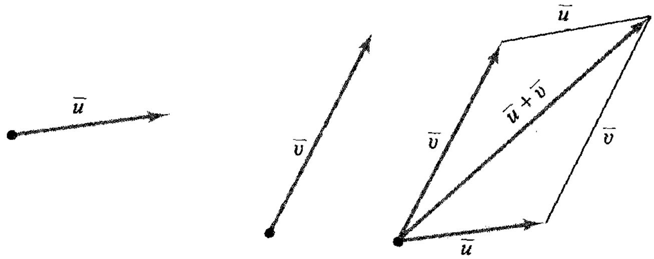


Figure 1.4 Addition of vectors (parallelogram rule)

The **length** or **magnitude** of $\bar{v}$ will be denoted by $\|\bar{v}\|$. The **direction** of $\bar{v}$ will be denoted by dir $\bar{v}$. If $\bar{u}$ and $\bar{v}$ are vectors such that the angle between them is 180° when they are drawn from the same point, then we shall write dir $\bar{u} = -$dir $\bar{v}$ and read "the direction of $\bar{u}$ is opposite the direction of $\bar{v}$" (Figure 1.5).

The **zero vector**, denoted by $\bar{0}$, is a vector of 0 length. We do not assign a direction to $\bar{0}$. Note that

$$\bar{v}+\bar{0}=\bar{0}+\bar{v}=\bar{v}$$

by the triangle rule for addition.

For any vector $\bar{v}$ we define $-\bar{v}$ (read "minus v" or "the opposite of v" or "the **additive inverse** of $\bar{v}$"; see Figure 1.6) to be the vector such that

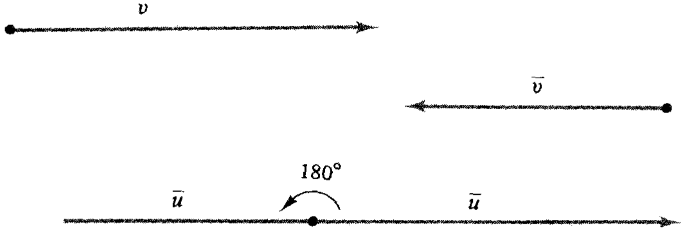


Figure 1.5 Dir $\bar{u} = -$dir $\bar{v}$

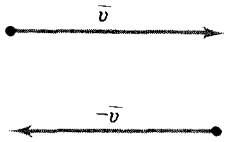


Figure 1.6 The additive inverse of $\bar{v}$

$$\|-\bar{v}\| = \|\bar{v}\|$$

and

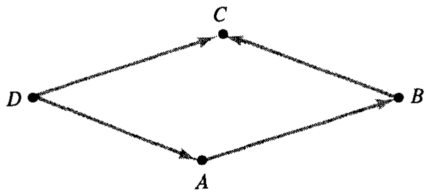
$$\text{dir } -\bar{v} = -\text{dir } \bar{v}$$

(If $\bar{v} = \bar{0}$, then $-\bar{0} = \bar{0}$.) From the triangle rule for addition it is clear that

$$\bar{v} + -\bar{v} = \bar{0}$$

If line vectors $\bar{u}$ and $\bar{v}$ have representations that are equal in length and parallel, then either $\bar{u} = \bar{v}$ or $\bar{u} = -\bar{v}$. *We shall assume that we can tell from the directions that the arrows point whether or not $\bar{u} = \bar{v}$ or $\bar{u} = -\bar{v}$.*

Example 1.1. In the parallelogram below, decide (a) which arrows represent the same vector, and (b) which arrows represent opposite vectors.



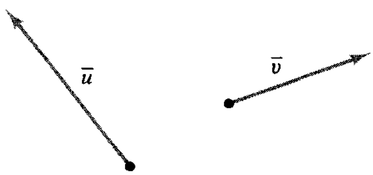
Solution: (a) ***Line segments*** $\overline{AB}$ and $\overline{DC}$ are opposite sides of the parallelogram. Therefore, $\|\overrightarrow{AB}\| = \|\overrightarrow{DC}\|$ and $\overline{AB}$ and $\overline{DC}$ are parallel. From the figure, dir $\overrightarrow{AB}$ = dir $\overrightarrow{DC}$. Therefore, $\overrightarrow{AB} = \overrightarrow{DC}$.

(b) Line segments $\overline{BC}$ and $\overline{DA}$ are opposite sides of the parallelogram. So $\overline{BC}$ and $\overline{DA}$ are parallel and $\|\overrightarrow{BC}\| = \|\overrightarrow{DA}\|$. From the figure, dir $\overrightarrow{BC}$ = $-$dir $\overrightarrow{DA}$. Hence $\overrightarrow{BC} = -(\overrightarrow{DA})$.

Since every line vector has an additive inverse, we can define **subtraction of vectors** by

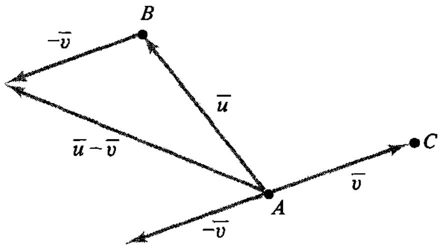
$$\bar{u} - \bar{v} = \bar{u} + (-\bar{v})$$

Example 1.2. Given representations for $\bar{u}$ and $\bar{v}$, use the definition of subtraction and the triangle rule for addition to draw $\bar{u} - \bar{v}$.



Solution

1. Draw $\bar{u} = \overrightarrow{AB}$ and $\bar{v} = \overrightarrow{AC}$ from the same point A.

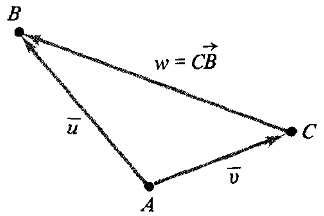


2. Draw $-\bar{v}$.
3. Using the triangle rule for addition, draw $\bar{u} - \bar{v} = \bar{u} + (-\bar{v})$.

Example 1.3. Show that $\bar{u} - \bar{v}$ **is the vector which when added to** $\bar{v}$ **is** $\bar{u}$.

Solution

1. Draw representations for $\bar{u}$ and $\bar{v}$ from some common point A. Let $\bar{u} = \overrightarrow{AB}$ and $\bar{v} = \overrightarrow{AC}$.



2. Complete the triangle by drawing $\bar{w} = \overrightarrow{CB}$.

By the triangle rule for addition, $\bar{v} + \bar{w} = \bar{u}$. Writing $\bar{v} + \bar{w}$ as $\bar{w} + \bar{v}$ and adding $-\bar{v}$ to both sides, we have $(\bar{w} + \bar{v}) + (-\bar{v}) = \bar{u} + (-\bar{v})$. Reassociating parentheses (Problem 2, Exercise 1.1) and using the definition of subtraction, we have

$$\bar{w} + (\bar{v} + (-\bar{v})) = \bar{u} - \bar{v}$$

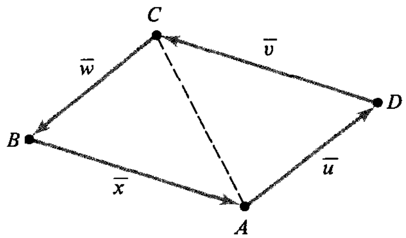
Since $\bar{v} + (-\bar{v}) = \bar{0}$ and $\bar{w} + \bar{0} = \bar{w}$, we get $\bar{w} = \bar{u} - \bar{v}$.

Note: The reader should compare the result of Example 1.2 with the result of Example 1.3.

Exercise 1.1

In the problems that call for drawings the reader will find that a transparent ruler is convenient for drawing parallel segments.

1. Let $\bar{u} = \overrightarrow{AD}$, $\bar{v} = \overrightarrow{DC}$, $\bar{w} = \overrightarrow{CB}$, and $\bar{x} = \overrightarrow{BA}$, as the figure indicates. Assume that $ABCD$ is a parallelogram with $\overline{AD}$ parallel to $\overline{BC}$.



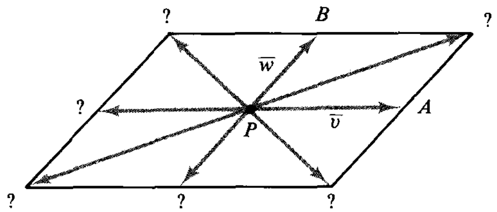
(a) Explain why $\overrightarrow{AD} = \overrightarrow{BC}$.
(b) Explain why $\bar{x} = -\bar{v}$.
(c) Write $\overrightarrow{AC}$ in terms of $\bar{u}$ and $\bar{v}$.
(d) Write $\overrightarrow{AC}$ in terms of $\bar{x}$ and $\bar{w}$.
(e) Find the sums $\bar{x} + \bar{u}$, $(\bar{x} + \bar{u}) + \bar{v}$, and $((\bar{x} + \bar{u}) + \bar{v}) + \bar{w}$, and express the answers in terms of the vertices of the parallelogram.

2. Show geometrically that vector addition is associative; that is,

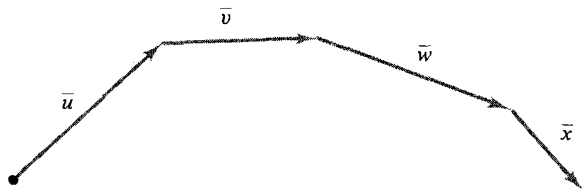
$$(\bar{u} + \bar{v}) + \bar{w} = \bar{u} + (\bar{v} + \bar{w})$$

Hint: Use the triangle rule for addition.

3. Label the unlabeled arrows in the figure below in terms of $\bar{v} = \overrightarrow{PA}$ and $\bar{w} = \overrightarrow{PB}$. You may assume that segments which look parallel are parallel. All arrows have their tails at the point P in the center of the parallelogram.



4. (a) In the figure below find $((\bar{u} + \bar{v}) + \bar{w}) + \bar{x}$ and $\bar{u} + (\bar{v} + (\bar{w} + \bar{x}))$.

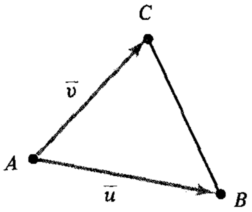


(b) Would it be appropriate to label the vector from the tail of $\bar{u}$ to the head of $\bar{x}$ as $\bar{u} + \bar{v} + \bar{w} + \bar{x}$? Why?

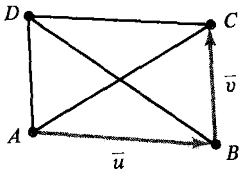
5. In each figure two of the vectors are labeled $\bar{u}$ and $\bar{v}$. Use these labels to label

each of the vectors called for. You may assume that segments that look parallel are parallel.

(a) $\overrightarrow{BC} =$ _____
$\overrightarrow{CB} =$ _____
$\overrightarrow{CA} =$ _____



(b) $\overrightarrow{CD} =$ _____
$\overrightarrow{DA} =$ _____
$\overrightarrow{DB} =$ _____
$\overrightarrow{AC} =$ _____



6. (a) If dir $\bar{v} = -$dir $\bar{u}$ and dir $(\bar{v} + \bar{u}) =$ dir $\bar{v}$, how does $\|\bar{v}\|$ compare with $\|\bar{u}\|$?
(b) If $\|\bar{v}\| < \|\bar{u}\|$ and dir $\bar{v} = -$dir $\bar{u}$, what is dir $(\bar{u} + \bar{v})$? (Give your answer in terms of either dir $\bar{u}$ or dir $\bar{v}$.)

7. Given line vectors $\bar{u}$, $\bar{v}$, and $\bar{w}$ as in Figure 1.7. Draw (a) $\bar{u} + \bar{v}$; (b) $(\bar{u} + \bar{v}) + \bar{w}$; (c) $\bar{v} + \bar{w}$; (d) $\bar{u} + (\bar{v} + \bar{w})$.

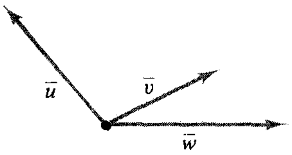


Figure 1.7

8. For line vectors $\bar{u}, \bar{v}$, and $\bar{w}$ as in Figure 1.7, draw (a) $\bar{u} - \bar{v}$; (b) $\bar{v} - \bar{w}$; (c) $\bar{u} - \bar{w}$; (d) how is the line vector in (c) related to the vectors in (a) and (b)?

9. Draw a triangle with sides $\bar{a}$, $\bar{b}$, and $\bar{a} + \bar{b}$.
(a) Why is $\|\bar{a} + \bar{b}\| \leq \|\bar{a}\| + \|\bar{b}\|$?
(b) For arbitrary line vectors $\bar{a}$ and $\bar{b}$, under what condition(s) is it true that $\|\bar{a} + \bar{b}\| = \|\bar{a}\| + \|\bar{b}\|$?

10. Draw a triangle with sides $\bar{a}$, $\bar{b}$, and $\bar{a} - \bar{b}$.
(a) Why is $|\|\bar{a}\| - \|\bar{b}\|| \leq \|\bar{a} - \bar{b}\|$? (The outside vertical lines on the left side of the inequality denote absolute value.)
(b) For arbitrary line vectors $\bar{a}$ and $\bar{b}$, under what condition(s) is it true that $\|\bar{a}\| - \|\bar{b}\| = \|\bar{a} - \bar{b}\|$?

11. Verify that $-(-\bar{v}) = \bar{v}$ by showing that the two vectors have the same length and the same direction.