Lecture Notes in Mathematics

Catherine Donati-Martin
Michel Émery
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Christophe Stricker (Eds.)

Séminaire de Probabilités XL

1899

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Lecture Notes in Mathematics

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Editorial Policy for Multi-Author Publications: Summer Schools / Intensive Courses

1. Lecture Notes aim to report new developments in all areas of mathematics and their applications - quickly, informally and at a high level. Mathematical texts analysing new developments in modelling and numerical simulation are welcome. Manuscripts should be reasonably self-contained and rounded off. Thus they may, and often will, present not only results of the author but also related work by other people. They should provide sufficient motivation, examples and applications. There should also be an introduction making the text comprehensible to a wider audience. This clearly distinguishes Lecture Notes from journal articles or technical reports which normally are very concise. Articles intended for a journal but too long to be accepted by most journals, usually do not have this „lecture notes" character.

2. In general SUMMER SCHOOLS and other similar INTENSIVE COURSES are held to present mathematical topics that are close to the frontiers of recent research to an audience at the beginning or intermediate graduate level, who may want to continue with this area of work, for a thesis or later. This makes demands on the didactic aspects of the presentation. Because the subjects of such schools are advanced, there often exists no textbook, and so ideally, the publication resulting from such a school could be a first approximation to such a textbook.

   Usually several authors are involved in the writing, so it is not always simple to obtain a unified approach to the presentation.

   For prospective publication in LNM, the resulting manuscript should not be just a collection of course notes, each of which has been developed by an individual author with little or no co-ordination with the others, and with little or no common concept. The subject matter should dictate the structure of the book, and the authorship of each part or chapter should take secondary importance. Of course the choice of authors is crucial to the quality of the material at the school and in the book, and the intention here is not to belittle their impact, but simply to say that the book should be planned to be written by these authors jointly, and not just assembled as a result of what these authors happen to submit.

   This represents considerable preparatory work (as it is imperative to ensure that the authors know these criteria before they invest work on a manuscript), and also considerable editing work afterwards, to get the book into final shape. Still it is the form that holds the most promise of a successful book that will be used by its intended audience, rather than yet another volume of proceedings for the library shelf.

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   Volume editors and authors should be aware that incomplete or insufficiently close to final manuscripts almost always result in longer evaluation times. They should also be aware that parallel submission of their manuscript to another publisher while under consideration for LNM will in general lead to immediate rejection.

Continued on inside back-cover

Lecture Notes in Mathematics 1899

While correcting the proofs of this volume, we received the sad news that Frank Knight had passed away. His deep understanding of stochastic processes, and in particular of their local times, has inspired many an author in the Séminaire de Probabilités; his contributions stand as models for clarity and rigor. The Séminaire has lost a very close friend and contributor.

This volume is dedicated to his memory.

C. Donati-Martin
M. Émery
A. Rouault
C. Stricker
M. Yor

Frank Knight (1933–2007): An appreciation, with respect and admiration

by Marc Yor

In March 2007, Frank passed away after a long illness. He is well known for having extracted some gems in the world of diffusions, e.g., the famous Ray–Knight theorems on local times, to mention one of his celebrated achievements.

Once started on a research topic, he was possessed by a very strong drive to solve the problem in a rigorous way: the reader of these lines should take the opportunity to have a look at his *Impressions of P.A. Meyer as Deus Ex-Machina* [1], where most of the discussion consists in disentangling some flaws in Frank's paper [2], for which P.A. Meyer helped him very generously...

Many exchanges with Frank were of this kind, as he wrote letters about fine points of martingale time changes [3], discussed the Krein theorem in relation with inverse local times [4], was interested in the Brownian spider [5], developed his beloved Prediction Theory ([6], [7]), and so on.

It is a tautology to say that Frank Knight had his own way of looking at things; see e.g., the Foreword to his *Essentials of Brownian Motion and Diffusion* [8] where he explains why no stochastic integrals will be found in the book ...

Despite his illness, he worked until the end, as shown by his joint paper [9] published in the volume [10] edited by D. Burkholder in memory of J. Doob.

However, a few months after P.A. Meyer's death, when I asked him to be part of the scientific committee for the Memorial Conference in February 2004 in Strasbourg, Frank wrote kindly that I was not being "reasonable"...

This was typical of Frank's seriousness, often mingled with humor.

I feel, as many of the Séminaire's oldies, that a great probabilist just started off his ultimate random walk.

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Preface

Who could have predicted that the *Séminaire* de Probabilités would reach the age of 40? This long life is first due to the vitality of the French probabilistic school, for which the *Séminaire* remains one of the most specific media of exchange. Another factor is the amount of enthusiasm, energy and time invested year after year by the Rédacteurs: Michel Ledoux dedicated himself to this task up to Volume XXXVIII, and Marc Yor made his name inseparable from the *Séminaire* by devoting himself to it during a quarter of a century. Browsing among the past volumes can only give a faint glimpse of how much is owed to them; keeping up with the standard they have set is a challenge to the new Rédaction.

In a changing world where the status of paper and ink is questioned and where, alas, pressure for publishing is increasing, in particular among young mathematicians, we shall try and keep the same direction. Although most contributions are anonymously refereed, the *Séminaire* is not a mathematical journal; our first criterion is not mathematical depth, but usefulness to the French and international probabilistic community. We do not insist that everything published in these volumes should have reached its final form or be original, and acceptance–rejection may not be decided on purely scientific grounds. The policy set forth in volume XIII still prevails: "laisser une place aux débutants à côté des mathématiciens déjà connus, publier des articles de mise au point à côté des travaux originaux, et même, de temps en temps, publier un article intéressant, mais faux."

But the *Séminaire* is not gray literature either. Most of its content, from the very beginning, is still interesting; we hope the current volumes will still be read many years from now. The advanced courses, started in volume XXXIII, are continued in this volume with Laure Coutin's account of calculus for fractional Brownian motion. The *Séminaire* also occasionally publishes a series of contributions on some given theme; in this spirit, a few participants to a May 2004 Oberwolfach workshop on local time-space calculus are contributing to the present volume, and the reports of their interventions give an overview on the current state of that subject.

For our 40th anniversary, Mathdoc has made us an invaluable present, for which we are very thankful: in the framework of their NUMDAM programme, the whole collection of *Séminaires de Probabilités* up to volume XXXVI has been digitized. The result is made available on `http://www.numdam.org/`; access to volumes I–XXXV is free, but, for the time being, access to volume XXXVI in only possible for subscribers to the Springer *link*.

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Part III Other Contributions

Part I

Specialized Course

An Introduction to (Stochastic) Calculus with Respect to Fractional Brownian Motion

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Summary. This survey presents three approaches to (stochastic) integration with respect to fractional Brownian motion. The first, a completely deterministic one, is the Young integral and its extension given by rough path theory; the second one is the extended Stratonovich integral introduced by Russo and Vallois; the third one is the divergence operator. For each type of integral, a change of variable formula or Itô formula is proved. Some existence and uniqueness results for differential equations driven by fractional Brownian motion are available except for the divergence integral. As soon as possible, these integrals are compared.

Key words: Gaussian processes, Fractional Brownian motion, Rough path, Stochastic calculus of variations

1 Introduction

Fractional Brownian motion was originally defined and studied by Kolmogorov, [Kol40] within a Hilbert space framework. Fractional Brownian motion of Hurst index $H \in]0,1[$ is a centered Gaussian process W^H with covariance function

$$\mathbf{E}\left(W^H(t)W^H(s)\right) = \frac{1}{2}\left[t^{2H} + s^{2H} - |t-s|^{2H}\right]; \quad (s,t \geqslant 0)$$

(for $H = \frac{1}{2}$, $W^{\frac{1}{2}}$ is a Brownian motion). Fractional Brownian motion has stationary increments since

$$\mathbf{E}\left[\left(W^H(t) - W^H(s)\right)^2\right] = |t-s|^{2H} \quad (s,t \geqslant 0)$$

and is H-selfsimilar:

$$\left(\frac{1}{c^H}W^H(ct); \quad t \geqslant 0\right) \stackrel{d}{=} \left(W^H(t); \quad t \geqslant 0\right) \quad \text{for any} \quad c > 0.$$

The Hurst parameter H accounts not only for the sign of the correlation of the increments, but also for the regularity of the sample paths. Indeed, for $H > \frac{1}{2}$, the increments are positively correlated, and for $H < \frac{1}{2}$, they are negatively correlated. Futhermore, for every $\beta \in (0, H)$, the sample paths are almost surely Hölder continuous with index β. Finally, for $H > \frac{1}{2}$, according to Beran's definition [BT99], it is a long memory process; the covariance of the increments at distance u decreases as u^{2H-2}.

These significant properties make fractional Brownian motion a natural candidate as a model for noise in mathematical finance (see Comte and Renault [CR96], Rogers [Rog97], Cheridito, [Che03], and Duncan, [Dun04]); in hydrology (see Hurst, [Hur51]), in communication networks (see, for instance Leland, Taqqu, Wilson, and Willinger [WLW94]). It appears in other fields, for instance, fractional Brownian motion with Hurst parameter $\frac{1}{4}$ is the limit process of the position of a particule in a one-dimensional nearest neighbor model with a convenient renormalization, see Rost and Vares [RV85]. For more applications, the reader should look at the monograph of Doukhan et al., [DOT03].

For $H \neq \frac{1}{2}$, W^H is neither a semimartingale (see, e.g., Example 2 of Section 4.9.13 of Liptser and Shiryaev [LS84]), nor a Markov process, and the usual Itô stochastic calculus does not apply. Our aim is to present different possible definitions of an integral

$$\int_0^t a(s)\, dW^H(s) \tag{1}$$

for a a suitable process and W^H a fractional Brownian motion, such that:

- The link with the Riemann sums is as expected: for a regular enough,

$$\lim_{|\pi| \to 0} \sum_{t_i \in \pi} a(t_i) \left(W^H(t_{i+1}) - W^H(t_i) \right) = \int_0^t a(s)\, dW^H(s)$$

  where $\pi = (t_i)_{i=0}^n$ are subdivisions of $[0, 1]$;
- There exists a change of variable formula, that is, for suitable f,

$$f\left(W^H(t)\right) = f(0) + \int_0^t f'\left(W^H(s)\right)\, dW^H(s);$$

- It allows to define and solve differential equations driven by a d-dimensional fractional Brownian motion $W^H = (W^i)_{i=1,\ldots,d}$,

$$y^i(t) = y^i(0) + \int_0^t f_0^i(y(s))\, ds + \sum_{j=1}^d \int_0^t f_j^i(y(s))\, dW^j(s),$$

where $y^i(0) \in \mathbb{R}$ and the functions f_j^i are smooth enough ($i = 1, \ldots, n$ and $j = 0, \ldots, d$).

The dimension is important. Assume that it is equal to one, $d = 1 = n$, and follow some ideas of Föllmer [Föl81]. Let f be a function, m times continuously differentiable. The sample paths of W^H are Hölder continuous of any index strictly less than H. Then, using a Taylor expansion of order $m > \frac{1}{H}$, the following limit exists

$$\int_0^t f'\left(W^H(s)\right) dW^H(s) := \lim_{|\pi| \to 0} \sum_{t_i \in \pi} \sum_{k=1}^{m} \frac{f^{(k)}\left(W^H(t_i)\right)\left(W^H(t_{i+1}) - W^H(t_i)\right)^k}{k!}$$

and $\int_0^t f'\left(W^H(s)\right) dW^H(s) = f\left(W^H(t)\right)$. When $d > 1$, one can also define

$$\int_0^t f\left(W^H(s)\right) dW^H(s) :=$$
$$= \lim_{|\pi| \to 0} \sum_{t_i \in \pi} \sum_{k=1}^{m} \frac{1}{k!} D^k F\left(W^H(t_i)\right) . \left(W^H(t_{i+1}) - W^H(t_i)\right)^{\otimes k}$$

if $f = DF$ with $F : \mathbb{R}^d \to \mathbb{R}$. Moreover, the ideas of Doss, [Dos77], allow to define and solve differential equations driven by a fractional Brownian motion, as proved in [Nou05]. This method extends to the multidimensional case when the Lie algebra generated by $f_1, \ldots, f_d$ is nilpotent, see Yamato [Yam79] for $H = 1/2$. This is pointed out in [BC05b].

These two arguments do not work in the multidimensional case in more general situations. The aim of this survey is to present other integrals which allow to work in dimension greater than one.

Recently, there have been numerous attempts to define a (stochastic) integral with respect to fractional Brownian motion.

- The first kind of attempts are deterministic ones. They rely on the properties of the sample paths. First, the results of Young, [You36], apply to fractional Brownian motion. The sample paths are Hölder continuous of any index strictly less than H. Then, the sequences of Riemann sums converge for any process a with sample paths α Hölder continuous with $\alpha + H > 1$. Secondly, Ciesielski et al. [CKR93] have noticed that the sample paths belong to some Besov–Orlicz space. They define an integral on Besov–Orlicz using wavelet expansions. Third, Zähle, [Zäh98] uses fractional calculus and a generalization of the integration by parts formula.
  For all these integrals, the process $f(W^H)$, for suitable functions f is integrable with respect to itself only if $H > \frac{1}{2}$. For $H > \frac{1}{2}$, there exists