GENERAL RELATIVITY

An introduction to the theory of the gravitational field

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Foreword

This book is designed to provide an introduction to the foundations of Einstein's theory and also to survey the questions it raises, its concepts and its methods. Because of the rapid development of relativity physics in recent years it has been impossible to avoid some restriction and selection of the subject matter; it is hoped, however, that the gap can be filled as far as possible by means of the bibliography at the end of every major section. Several rather more exacting sections, which the reader may omit at a first reading, are denoted by an asterisk. The reader is assumed to be familiar with theoretical mechanics, electrodynamics and special relativity. The basic ideas of Riemannian geometry which are necessary for the theory of general relativity are described in the first chapters.

My thanks go to all colleagues of the Jena research group, led by Professor Schmutzer, with whom and from whom I have learnt the theory of relativity, and to all authors of books and articles, whether mentioned by name or not, whose ideas this book contains. I am especially indebted to my colleagues Dr. G. Kluge and Dr. D. Kramer for numerous critical remarks on the form of the manuscript, and to Professor Dr. E. Schmutzer and Dr. F. Gehlhar who suggested changes. I also have to thank Frau U. Kaschlik for a careful typing of the manuscript, Herr Th. Elster for help with the corrections, and not least the publisher for a pleasant collaboration.

HANS STEPHANI

Jena 1977

Editor's Preface

Several years ago I was asked by an astrophysicist to recommend a textbook on relativity which, although physically oriented, contained a clear, unbiassed description of mainstream relativity. On the same day Cambridge University Press sent me Dr Hans Stephani's Allgemeine Relativitätstheorie to review, thus answering the astrophysicist's question. This book is a translation of the first (1977) edition with all of the amendments and corrections of the second (1980) edition included. In a few places I have added comments in footnotes. Further, the bibliography has been updated to 1980/81, and where possible English translations have been cited.

JOHN STEWART

Notations, conventions and important formulae

Minkowski space: $ds^2 = \eta_{ab}dx^a dx^b = dx^2 + dy^2 + dz^2 - c^2dt^2$.

Riemannian space: $ds^2 = g_{ab}dx^a dx^b = -c^2 d\tau^2$;

$$g = |g_{ab}|, \quad g^{ab}g_{bm} = \delta^a_m = g^a_m.$$

ε-pseudo-tensor: ε^{abmn}; $\varepsilon^{1234} = 1/\sqrt{-g}$,

$$\varepsilon_{abcd}\varepsilon^{abnm} = -2(g^n_c g^m_d - g^m_c g^n_d).$$

Dualisation of an antisymmetric tensor: $\tilde{F}^{ab} = \frac{1}{2}\varepsilon^{abmn}F_{mn}$.

Christoffel symbols: $\Gamma^a_{mn} = \frac{1}{2}g^{ab}(g_{bm,n} + g_{bn,m} - g_{mn,b})$.

Covariant derivative: $DT^a/Dx^m = T^a{}_{;m} = T^a{}_{,m} + \Gamma^a_{mn}T^n$,

$$DT_a/Dx^m = T_{a;m} = T_{a,m} - \Gamma^n_{am}T_n.$$

Geodesic equation: $\dfrac{D^2x^i}{D\lambda^2} = \dfrac{d^2x^i}{d\lambda^2} + \Gamma^i_{nm}\dfrac{dx^n}{d\lambda}\dfrac{dx^m}{d\lambda} = 0.$

Parallel transport along the curve $x^i(\lambda)$: $DT^a/D\lambda = T^a{}_{;b}dx^b/d\lambda = 0$.

Fermi–Walker transport: $\dfrac{DT^n}{D\tau} - \dfrac{1}{c^2}T_a\left(\dfrac{dx^n}{d\tau}\dfrac{D^2x^a}{D\tau^2} - \dfrac{dx^a}{d\tau}\dfrac{D^2x^n}{D\tau^2}\right) = 0.$

Lie derivative in the direction of the vector field $a^k(x^i)$:

$$\underset{a}{£}T^n = T^n{}_{,k}a^k - T^k a^n{}_{,k} = T^n{}_{;k}a^k - T^k a^n{}_{;k}.$$

$$\underset{a}{£}T_n = T_{n,k}a^k + T_k a^k{}_{,n} = T_{n;k}a^k + T_k a^k{}_{;n}.$$

Killing equation: $\xi_{i;n} + \xi_{n;i} = \underset{\xi}{£}g_{in} = 0$.

Divergence of a vector field: $a^i{}_{;i} = \dfrac{1}{\sqrt{-g}}(\sqrt{-g}\,a^i)_{,i}$

Maxwell equations: $F^{mn}{}_{;n} = \dfrac{1}{\sqrt{-g}}(\sqrt{-g}\,F^{mn})_{,n} = \dfrac{1}{c}j^m$,

$$\tilde{F}^{mn}{}_{;n} = 0.$$

Curvature tensor: $a_{m;s;q} - a_{m;q;s} = a_b R^b{}_{msq}$,

$$R^b{}_{msq} = \Gamma^b_{mq,s} - \Gamma^b_{ms,q} + \Gamma^b_{ns}\Gamma^n_{mq} - \Gamma^b_{nq}\Gamma^n_{ms},$$

$$R_{amsq} = \tfrac{1}{2}(g_{aq,ms} + g_{ms,aq} - g_{as,mq} - g_{mq,as}) + \text{non-linear terms.}$$

Ricci tensor: $R_{mq} = R^s{}_{msq} = -R^s{}_{mqs}$; $\quad R^m{}_m = R$.

Field equations: $G_{ab} = R_{ab} - \dfrac{R}{2} g_{ab} = \kappa T_{ab}$.

Perfect fluid: $T_{ab} = (\mu + p/c^2)u_a u_b + p g_{ab}$.

Schwarzschild metric: $$\mathrm{d}s^2 = \frac{\mathrm{d}r^2}{1 - 2M/r} + r^2(\mathrm{d}\vartheta^2 + \sin^2\vartheta\,\mathrm{d}\varphi^2) - (1 - 2M/r)c^2\mathrm{d}t^2.$$

Robertson–Walker metric: $$\mathrm{d}s^2 = K^2(ct) \times \left[\frac{\mathrm{d}r^2}{1 - \varepsilon r^2} + r^2(\mathrm{d}\vartheta^2 + \sin^2\vartheta\,\mathrm{d}\varphi^2)\right] - c^2\mathrm{d}t^2.$$

Hubble constant: $H(ct) = \dot{K}/K$.

Acceleration parameter: $q(ct) = -K\ddot{K}/K^2$.

$\kappa = 2.07 \times 10^{-48}\,\mathrm{g}^{-1}\,\mathrm{cm}^{-1}\,\mathrm{s}^2$, $\quad cH = 55$ km/s Mpc,

$2M_{\mathrm{Earth}} = 0.8876$ cm, $\quad 2M_{\mathrm{Sun}} = 2.9533 \times 10^5$ cm.

Contents

Introduction

The general theory of relativity came into being historically as an extension of the special theory of relativity. However the questions to which it is addressed are somewhat incompletely described by its name. As in every living, evolving science there is no agreement amongst scientists as to which of the viewpoints and ideas discussed were particularly important when the theory was in its infancy, which are absolutely necessary for a logical structure, or which will prove fruitful for its future development. Three groups of questions, however, played a special rôle and led ultimately to the general theory of relativity.

(1) For a description of nature and its laws one should be able to use *arbitrary coordinate systems*, and in accordance with the *principle of covariance* the form of the laws of nature should not depend essentially upon the choice of the coordinate system. This requirement, in the first place purely mathematical, acquires a physical meaning through the substitution of 'arbitrary coordinate system' by 'arbitrarily moving observer'. The laws of nature should be independent of the state of motion of the observer, as, analogously, they are the same in the special theory of relativity for all inertial systems, that is for all observers moving with constant speed relative to one another. To this group belongs also the question, raised in particular by Ernst Mach, of whether an absolute acceleration (including an absolute rotation) can really be defined meaningfully, or whether every measurable rotation implies a rotation relative to the fixed stars (*Mach's Principle*).

(2) The Newtonian theory of gravitation is inconsistent with the special principle of relativity. In it gravitational effects propagate with an infinitely large velocity. A new, better formulation of the *field equations of gravitation* should therefore be found, which includes also the influence of gravity upon other physical processes and which agrees with experiment.

(3) In astrophysics and *cosmology* large masses are involved; gravitational forces dominate short-range nuclear forces. Thus a theory of gravitation has to be found which correctly reflects the dynamical behaviour of the whole Universe and which at the same time is valid for stellar evolution.

The general theory of relativity began with the formulation of the fundamental equations by Albert Einstein in 1915, followed by a series of articles on the foundations of the theory and on its possible experimental confirmation. In spite of the success of the theory (precession of the perihelion of Mercury, deflection of light by the Sun, explanation of the cosmological redshift), it has retained for a long time the reputation of an esoteric science for specialists and outsiders, perhaps because of the mathematical difficulties, the new concepts and the paucity of applications (for example in comparison with quantum theory, which came into existence at almost the same time). Through the development of new methods of obtaining solutions and the physical interpretation of the theory, and even more through the surprising astrophysical discoveries (pulsars, cosmic background radiation), and the improved possibilities of demonstrating general relativistic effects, in the course of the last twenty years the general theory of relativity has become a true physical science, with many associated experimental questions and observable consequences.

The general theory of relativity is the theory of the gravitational field; the description of its language and concepts, and its methods and conclusions, form the main content of this book.

Modern theoretical physics uses and needs ever more complicated mathematical tools – this statement, with its often unwelcome consequences for the physicist, is true also for the theory of gravitation. The language of the general theory of relativity is differential geometry, and we must learn it, if we wish to ask and answer precisely physical questions. This book therefore begins with seven chapters in which the essential concepts and formulae of Riemannian geometry are described. As far as possible, mathematical discussions are physically motivated, sometimes by the use of concepts and ideas which can not be introduced precisely or really understood until later.

1 The force-free motion of particles in Newtonian mechanics

1.1 *Coordinate systems*

In theoretical mechanics one usually meets only a few simple coordinate systems for describing the motion of a particle. For the purposes of mechanics one can characterise the coordinate system best via the specification of the connection between the infinitesimal separation ds of two points and the difference of their coordinates. In describing the motion in three-dimensional space one chooses Cartesian coordinates x, y, z with

$$\mathrm{d}s^2 = \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2, \tag{1.1}$$

cylindrical coordinates ϱ, φ, z with

$$ds^2 = d\varrho^2 + \varrho^2 d\varphi^2 + dz^2 \tag{1.2}$$

or spherical coordinates r, ϑ, φ with

$$ds^2 = dr^2 + r^2 d\vartheta^2 + r^2 \sin^2 \vartheta \, d\varphi^2. \tag{1.3}$$

If the motion is restricted to a surface which does not change with time, for example a sphere, then one would use the corresponding two-dimensional section ($dr = 0$) of spherical coordinates

$$ds^2 = r^2 d\vartheta^2 + r^2 \sin^2 \vartheta \, d\varphi^2. \tag{1.4}$$

For other arbitrary coordinate systems ds^2 is also a quadratic function of the coordinate differentials:

$$ds^2 = g_{\alpha\beta}(x^\nu) dx^\alpha dx^\beta; \quad \alpha, \beta, \nu = 1, 2, 3. \tag{1.5}$$

Here and in all following formulae indices occurring twice are to be summed, from one to three for a particle in three-dimensional space and from one to two for a particle on a plane.

The form (1.5) is called the *fundamental metric form*; the position-dependent coefficients $g_{\alpha\beta}$ form the components of the *metric tensor*. It is symmetric: $g_{\alpha\beta} = g_{\beta\alpha}$. The name 'metric tensor' refers to the fact that by its use the quantities length and angle which are fundamental to geometrical measurement can be defined and calculated. The displacement ds of two points with coordinates (x^1, x^2) and $(x^1 + dx^1, x^2 + dx^2)$ is given by (1.5), and the angle ψ between two infinitesimal vectors $d^{(1)}x^\alpha$ and $d^{(2)}x^\alpha$ diverging from a point can be calculated as

$$\cos\psi = \frac{g_{\alpha\beta}\, d^{(1)}x^\alpha d^{(2)}x^\beta}{\sqrt{g_{\varrho\sigma}\, d^{(1)}x^\varrho d^{(1)}x^\sigma}\sqrt{g_{\mu\nu}\, d^{(2)}x^\mu d^{(2)}x^\nu}}. \tag{1.6}$$

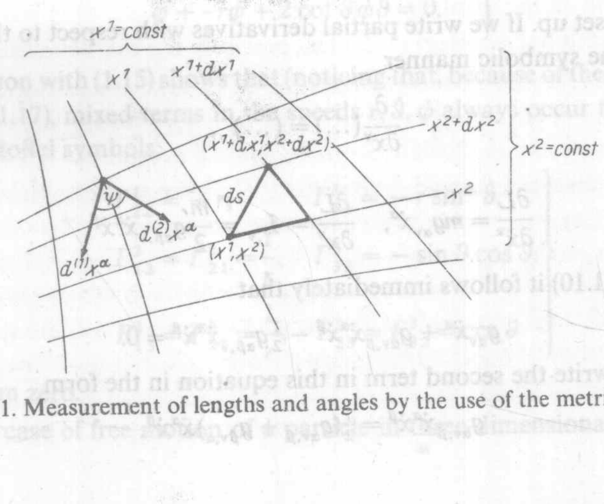

Fig. 1.1. Measurement of lengths and angles by the use of the metric tensor.

Formula (1.6) is nothing other than the familiar vector relation $ab = |a||b| \cos(a, b)$ applied to infinitesimal vectors.

If the matrix of the metric tensor is diagonal, that is to say $g_{\alpha\beta}$ differs from zero only when $\alpha = \beta$, then one calls the coordinate system orthogonal. As (1.6) shows, the coordinate lines x^α = constant are then mutually perpendicular.

If the determinant of $g_{\alpha\beta}$ is non-zero, the matrix possesses an inverse matrix $g^{\beta\mu}$ given by

$$g_{\alpha\beta}g^{\beta\mu} = \delta^\mu_\alpha = g^\mu_\alpha. \tag{1.7}$$

The immediate significance of the fundamental metric form (1.5) for mechanics rests on its simple connection with the square of the speed of the particle:

$$v^2 = \left(\frac{\mathrm{d}s}{\mathrm{d}t}\right)^2 = g_{\alpha\beta}\frac{\mathrm{d}x^\alpha}{\mathrm{d}t}\frac{\mathrm{d}x^\beta}{\mathrm{d}t}, \tag{1.8}$$

which we need for the construction of the kinetic energy as one part of the Lagrangian.

1.2. *Equations of motion*

We can obtain the equations of motion most quickly from the Lagrangian L, which for force-free motion is identical with the kinetic energy of the particle

$$L = \frac{m}{2}v^2 = \frac{m}{2}g_{\alpha\beta}\frac{\mathrm{d}x^\alpha \mathrm{d}x^\beta}{\mathrm{d}t \;\; \mathrm{d}t} = \frac{m}{2}g_{\alpha\beta}\dot{x}^\alpha\dot{x}^\beta. \tag{1.9}$$

The corresponding Lagrange equations (of the second kind)

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{x}^\nu} - \frac{\partial L}{\partial x^\nu} = 0 \tag{1.10}$$

are easily set up. If we write partial derivatives with respect to the coordinates in the symbolic manner

$$\frac{\partial}{\partial x^\nu}(\ldots) = (\ldots)_{,\nu}, \tag{1.11}$$

then

$$\frac{\partial L}{\partial \dot{x}^\nu} = mg_{\alpha\nu}\dot{x}^\alpha, \quad \frac{\partial L}{\partial x^\nu} = L_{,\nu} = \frac{m}{2}g_{\alpha\beta,\nu}\dot{x}^\alpha\dot{x}^\beta, \tag{1.12}$$

and from (1.10) it follows immediately that

$$g_{\alpha\nu}\ddot{x}^\alpha + g_{\alpha\nu,\beta}\dot{x}^\alpha\dot{x}^\beta - \tfrac{1}{2}g_{\alpha\beta,\nu}\dot{x}^\alpha\dot{x}^\beta = 0. \tag{1.13}$$

If we first write the second term in this equation in the form

$$g_{\alpha\nu,\beta}\dot{x}^\alpha\dot{x}^\beta = \tfrac{1}{2}(g_{\alpha\nu,\beta} + g_{\beta\nu,\alpha})\dot{x}^\alpha\dot{x}^\beta, \tag{1.14}$$

then multiply (1.13) by $g^{\mu\nu}$ and sum over ν, then because of (1.7) we obtain

$$\ddot{x}^\mu + \Gamma^\mu_{\alpha\beta}\dot{x}^\alpha\dot{x}^\beta = 0, \tag{1.15}$$

where the abbreviation

$$\Gamma^\mu_{\alpha\beta} = \tfrac{1}{2} g^{\mu\nu}(g_{\alpha\nu,\beta} + g_{\beta\nu,\alpha} - g_{\alpha\beta,\nu}) \tag{1.16}$$

has been used.

Equations (1.15) are the required equations of motion of a particle. In the course of their derivation we have also come across the *Christoffel symbols* $\Gamma^\mu_{\alpha\beta}$, defined by (1.16), which play a great rôle in differential geometry. As is evident from (1.16), they possess the symmetry

$$\Gamma^\mu_{\alpha\beta} = \Gamma^\mu_{\beta\alpha}, \tag{1.17}$$

and hence there are eighteen distinct Christoffel symbols in three-dimensional space, and six for two-dimensional surfaces.

On contemplating (1.15) and (1.16), one might suppose that the Christoffel symbols lead to a particularly simple way of constructing the equations of motion. This supposition is, however, false; on the contrary, one needs the very equations of motion in order to construct the Christoffel symbols. We shall illustrate this method by means of an example. In spherical coordinates (1.3), $x^1 = r, x^2 = \vartheta, x^3 = \varphi$, the Lagrangian

$$L = \frac{m}{2}(\dot{r}^2 + r^2\dot{\vartheta}^2 + r^2\sin^2\vartheta\dot{\varphi}^2) \tag{1.18}$$

implies the following Lagrange equations of the second kind:

$$\left.\begin{aligned} &\ddot{r} - r\dot{\vartheta}^2 - r\sin^2\vartheta\,\dot{\varphi}^2 = 0,\\ &\ddot{\vartheta} + \frac{2}{r}\dot{r}\dot{\vartheta} - \sin\vartheta\cos\vartheta\,\dot{\varphi}^2 = 0,\\ \text{and}\qquad &\ddot{\varphi} + \frac{2}{r}\dot{r}\dot{\varphi} + 2\cot\vartheta\,\dot{\varphi}\dot{\vartheta} = 0. \end{aligned}\right\} \tag{1.19}$$

Comparison with (1.15) shows that (noticing that, because of the symmetry relation (1.17), mixed terms in the speeds $\dot{r}, \dot{\vartheta}, \dot{\varphi}$ always occur twice) only the Christoffel symbols

$$\left.\begin{aligned} &\Gamma^1_{22} = -r, &&\Gamma^1_{33} = -r\sin^2\vartheta,\\ &\Gamma^2_{12} = \Gamma^2_{21} = \frac{1}{r}, &&\Gamma^2_{33} = -\sin\vartheta\cos\vartheta,\\ &\Gamma^3_{13} = \Gamma^3_{31} = \frac{1}{r}, &&\Gamma^3_{23} = \Gamma^3_{32} = \cot\vartheta \end{aligned}\right\} \tag{1.20}$$

differ from zero.

In the case of free motion of a particle in three-dimensional space the