Mathematics and Its Applications

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Degenerate Elliptic Equations

Kluwer Academic Publishers

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by

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KLUWER ACADEMIC PUBLISHERS
DORDRECHT / BOSTON / LONDON

Library of Congress Cataloging-in-Publication Data

Levendorskiĭ, Serge.
Degenerate elliptic equations / by Serge Levendorskii.
p. cm. -- (Mathematics and its applications ; v. 258)
Includes index.
ISBN 0-7923-2305-X
1. Differential equations, Elliptic. I. Title. II. Series: Mathematics and its applications (Kluwer Academic Publishers) ; v. 258.
QA377.L4156 1994
515'.353--dc20 93-13188

ISBN 0-7923-2305-X

Published by Kluwer Academic Publishers,
P.O. Box 17, 3300 AA Dordrecht, The Netherlands.

Kluwer Academic Publishers incorporates
the publishing programmes of
D. Reidel, Martinus Nijhoff, Dr W. Junk and MTP Press.

Sold and distributed in the U.S.A. and Canada
by Kluwer Academic Publishers,
101 Philip Drive, Norwell, MA 02061, U.S.A.

In all other countries, sold and distributed
by Kluwer Academic Publishers Group,
P.O. Box 322, 3300 AH Dordrecht, The Netherlands.

Printed on acid-free paper

Printed in the Netherlands

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Chapter 0

Introduction

0.1 The partial differential equation

$$(1) \qquad (Au)(x) = \sum_{|\alpha|\leq 2m} a_\alpha(x)(D^\alpha u)(x) = f(x)$$

is called elliptic on a set G, provided that the principal symbol

$$a_{2m}(x,\xi) = \sum_{|\alpha|=2m} a_\alpha(x)\xi^\alpha$$

of the operator A is invertible on $G \times (\mathbb{R}^n \setminus 0)$; A is called elliptic on G, too. This definition works for systems of equations, for classical pseudodifferential operators (ψdo), and for operators on a manifold Ω.

Let us recall some facts concerning elliptic operators.

If Ω is closed, then for any $s \in \mathbb{R}^1$,

$$A : H^s(\Omega; \mathbb{C}^v) \to H^{s-2m}(\Omega; \mathbb{C}^v)$$

is Fredholm and the following *a priori* estimate holds

$$(2) \qquad \| u \|_s \leq c(s)(\| Au \|_{s-2m} + \| u \|_{s-1}).$$

If $m > 0$ and $A : C^\infty(\Omega;\mathbb{C}^v) \to L_2(\Omega;\mathbb{C}^v)$ is formally self – adjoint with respect to a smooth positive density, then the closure A_0 of A is a self – adjoint operator with discrete spectrum and for the distribution functions of the positive and negative eigenvalues (counted with multiplicity) of A_0 one has the following Weyl formula: as $t \to \infty$,

$$(3) \qquad N_\pm(t, A_0) \sim (2\pi)^{-n} \iint_{T^*\Omega\setminus 0} N_\pm(t, a_{2m}(x,\xi))\, dx d\xi =$$

$$= t^{n/2m} \iint_{T^*\Omega\setminus 0} N_\pm(1, a_{2m}(x,\xi))\, dx d\xi$$

(on the right hand side, $N_\pm(t, a_{2m}(x,\xi))$are the distribution functions of the matrix $a_{2m}(x,\xi) : \mathbb{C}^v \to \mathbb{C}^v$).

We now assume that Ω is a compact manifold with the boundary $\Gamma = \partial\Omega$ of the class C^{2m} and A is an elliptic differential operator on Ω of order $2m$. Let us consider a boundary value problem of the form

$$(4) \qquad Au = f \quad \text{on } \Omega, \qquad B_j u = g_j \quad \text{on } \Gamma, \qquad 1 \le j \le m,$$

where B_j is a differential operator of order $m_j < 2m$ $(1 \le j \le m)$. The operator

$$\mathfrak{A} : C^\infty(\bar\Omega) \ni u \mapsto (Au, B_1 u|_\Gamma, \ldots, B_m u|_\gamma) \in$$
$$\in C^\infty(\bar\Omega) \oplus C^\infty(\Gamma;\mathbb{C}^m)$$

of the problem (4) is called elliptic, provided that both the principal symbol of A (now called the principal interior symbol of the boundary value problem (4) and operator $\mathfrak{A}$) and the principal boundary symbol of the boundary value problem (4) and operator $\mathfrak{A}$ are invertible. The latter is the family $a(x',\xi')$ $((x',\xi') \in T^*\Gamma \setminus 0)$of boundary value problems on half – axis. If the boundary value problem (4) is elliptic and $s > m_j + \frac{1}{2}$ for all j, then $\mathfrak{A}$ admits a unique continuous extension

$$\mathfrak{A}_s : H^s(\Omega) \to H^{s-2m}(\Omega) \bigoplus_{1\le j\le m} H^{s-m_j-\frac{1}{2}}(\Gamma).$$

$\mathfrak{A}_s$ is Fredholm and the following analogue of (2) holds

$$\| u \|_s \leq c(s)(\| Au \|_{s-2m} + \sum_{1 \leq j \leq m} \| B_j u|_\Gamma \|_{s-m_j-\frac{1}{2}} + \| u \|_{s-1}).$$

Let $A : C_0^\infty(\Omega; \mathbb{C}^v) \to L_2(\Omega; \mathbb{C}^v)$ be formally self – adjoint with respect to a smooth positive density and let A be a self – adjoint extension of A with the domain $D(A_0)$ such that

$$\overset{\circ}{H}{}^{2m}(\Omega, \mathbb{C}^v) \subset D(A_0) \subset H^{2m}(\Omega, \mathbb{C}^v).$$

Then (3) holds (in fact, essentially weaker conditions on Ω and $D(A_0)$ suffice).

The aim of the book is to study how all these statements change when the ellipticity condition fails.

For operators on closed manifold this means that a_{2m} is not invertible on a conical submanifold $\Sigma \subset T^*\Omega \setminus 0$, and for operators on a manifold with the boundary that either a_{2m} is not invertible on a conical submanifold $\Sigma \subset T^*\bar{\Omega}\setminus 0$ (usually, one considers the case $\Sigma = (T^*\bar{\Omega}\setminus 0)_{\Gamma_1}$ where $\Gamma_1 \subset \Gamma$, or the principal boundary symbol $\mathfrak{A}(x', \xi')$ is not invertible for some $(x', \xi') \in T^*\Gamma\setminus 0)$. In the book, we consider the first type of violation of the ellipticity condition; such operators and corresponding equations are called degenerate elliptic on Ω. Note that the studying of boundary value problems can be reduced to that of operators on Γ, the former being elliptic iff the latter are (see, for instance, Section 20.4 in Hörmander [7] and Section 4.2 in Rempel and Schulze [1]). Therefore, the results of the book imply some results on degenerate elliptic boundary value problems for elliptic operators as well.

Degenerate elliptic equations arise in the theory of shells, in the theory of Brownian motion and in many other problems of mathematical physics and mechanics. Changes of variables allow to reduce any equations in unbounded domains and in domains with singularities of the boundary to degenerate elliptic equations.

There are too many essentially different types of degeneration, therefore the universal classification of degenerate elliptic operators seems to be impossible. Nevertheless, in the case of sufficiently regular degeneration on the boundary of a bounded domain it appears to be possible to clarify which are the general properties that different classes of degenerate elliptic operators have in common and which of these classes are essentially different - both from the point of view of symbols describing operators and methods of investigation. This book is devoted to the study of such classes of differential operators and some related classes of ψdo.

0.2. The investigation of degenerate elliptic equations could be carried out along the same lines as that of elliptic equations (and many others) but the comprehensive investigation would make the book voluminous. We consider conditions for operator to be Fredholm in appropriate weighted Sobolev spaces, deduce *a priori* estimates of solutions, prove inequalities of Gårding type and compute the principal term of the spectral asymptotics for self - adjoint operators. In addition, for a class of boundary value problems for degenerate elliptic operators degenerating on the boundary into elliptic operators of different orders, the analogue of the Boutet de Monvel calculus (Boutet de Monvel [1], Rempel and Schutze [1]) is developed and the index is computed.

The conditions for operators to be Fredholm are formulated in terms of invertibility of the set of symbols associated with the operator under consideration. The type of this set determines the type of the operator. It is shown that the coefficients in the asymptotic formulae for distribution of eigenvalues are expressed in terms of these symbols.

One of the main distinctions between elliptic equations and degenerate elliptic ones is the dependence of the number and the type of boundary conditions on the character of degeneration. In the book, we distinguish the classes that do not require boundary conditions and consider the question about the number and the type of boundary conditions for operators of other classes. For degenerate elliptic operators on closed manifold, we consider the case when it is necessary to pose "interior boundary conditions" on the submanifold of degeneration. The latter type of degeneration concerns hypoelliptic pseudodifferential operators as well.

The computation of the spectral asymptotics is based on two general schemes, one of which (the formal scheme of computing of spectral asymptotics) plays the same part as the classical Weyl formula (3) does in regular situations.

0.3. We start in Section 1.1 with formulations of definitions and theorems of the Weyl - Hörmander calculus of ψdo in the form suggested in Levendorskiĭ [1]. Unlike the calculus in the book Hörmander [7], here the order of operator is determined by two operator - valued functions. The calculus of Section 1.1 serves as a model for construction of the calculus of ψdo with double symbols in Section 1.2. This calculus allows one to study differential operators in the case of the strong degeneration on the boundary by means of the methods of general theory of ψdo and to reduce the investigation of wide classes of degenerate operators to that of four model classes.

General classes are investigated in Chapters 3-5; model ones are investigated in Chapter 2, where rather simple methods are used. In particular, in order to be able to read Chapter 2, one only needs to be acquainted with the theory of elliptic ψdo on closed manifold and with that on $\mathbb{R}^n$.

In Chapter 6, the method of Chapter 3 is used to prove the Gårding inequality for degenerate elliptic quadratic form (or, equivalently, for degenerate elliptic differential operators in divergent form). We need this inequality to compute spectral asymptotics in Chapter 10.

In Chapter 7, ψdo of varying order on a closed manifold are considered. Hypoellipticity conditions are given and the dependence of solutions on smoothness of data is investigated. For the sake of brevity, we do not formulate local and microlocal results on hypoellipticity (cf., e.g., Chapter 22 in Hörmander [7]). Instead, we construct pairs of weighted Sobolev spaces in which operators under consideration are Fredholm. In some cases, when the hypoellipticity conditions fail, we obtain well

- posed problems by introducing "interior boundary value problems".

Chapter 8 shows that wide classes of degenerate elliptic operators possess almost all the properties of elliptic ones. In this Chapter, we define the class of ψdo of varying order with the transmission property and for these operators, we construct an algebra of boundary value problems. This algebras contains parametrices and inverses of operators of boundary value problems for classical ψdo that are hypoelliptic on Ω and strongly degenerate on the boundary into elliptic operators. The existence of such an algebra allows to compute indices of boundary value problems for degenerate elliptic operators (and investigate their functions, the resolvent being an example, but the volume of the book does not allow to do this).

Algebras of ψdo, related to degenerate elliptic operators of other types, are discussed in a brief review of bibliography.

Chapters 9 - 11 are devoted to the investigation of spectral asymptotics. Chapter 9 is auxiliary. In this Chapter, two - side estimates for the number of negative eigenvalues of an operator depending on parameter are obtained, and a general method based on these estimates, for computation of spectral asymptotics of wide classes of degenerate elliptic operators is given. The latter includes sets of formulas which plays the same part as the classical Weyl formula (3) does in regular situations.

In Chapter 10, spectral asymptotics for wide classes of degenerate elliptic differential operators are computed, and in Chapter 11, the same is done for self - adjoint hypoelliptic ψdo with multiple characteristics.

0.4. In the book, the use is made of general methods of the theory of ψdo. This allows to study equations of arbitrary order. Nevertheless, many results of the book had not been considered in the previous books even in the case of second order operators. These results include classification of the types of degeneration, the interior boundary value problems for hypoelliptic operators with multiple characteristics, the spectral asymptotics for all types of degenerate elliptic operators and hypoelliptic ψdo with multiple characteristics. The L_p-theory of boundary value problems for elliptic degenerate operators, the calculus of ψdo with the double symbols and the analogue of the Boutet de Monvel calculus are also new.

Some simplest classes of degenerate elliptic operators had been studied in Chapters 6,7 of the book Triebel [1] and operators of second order had been considered in book Oleinik and Radkevic [1] The results of Sections 7.2, 7.3 are generalizations of the well - known results on ψdo with double characteristics (see e.g., books Hörmander [7], Taylor [1]).

The results of Chapter 9 are essentially contained in book Levendorskiĭ [1] and so is a part of results of Subsection 10.2.4 (namely, the ones on isotropic degeneration). The latter book is devoted to spectral asymptotics but in it four types of degeneration are not distinguished, hypoelliptic ψdo with double characteristics are not investigated and conditions are formulated in terms of more complex symbols. On the other hand, in op.cit. more general model classes were studied.

0.5. There are three main circles of subjects studied in the book, namely:

1. Investigation of the Fredholm property, *a priori* estimates and analogues of Gårding inequality for wide classes of degenerate elliptic differential operators (Chapters 2-6);

2. The same for hypoelliptic ψdo (Chapters 7,8);

3. Computation of spectral asymptotics (Chapters 9-11).

Chapter 1 may be considered as an auxiliary one.

The kernel of the discussion of the first circle of problems is Chapter 2 where simple methods are used. Nevertheless, this Chapter allows to understand what the main properties of degenerate elliptic operators are and what are the main distinctions between various classes of these operators .