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Germs of Diffeomorphisms in the Plane

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Summary, some motivation and acknowledgments

The aim of this lecture note is to study germs of C^∞ diffeomorphisms in $\mathbb{R}^2$ from a topological and a C^∞ point of view (C^∞ means smooth or infinitely differentiable). Although our methods could also be used for a C^r study we do not pay attention to this here. We especially emphasize the following problems :

1. When can such a germ or a power of it be C^0 or C^∞ embedded in the germ of a flow ?
2. When are such germs C^∞ determined by their ∞ -jet ?
3. When are such germs C^0 determined by some finite jet ?

We restrict our attention to the germs occuring in generic n-parameter families of diffeomorphisms and having a characteristic line.

The possibility of embedding a diffeomorphism in a flow in a C^0 or C^∞ way (i.e. to show that the diffeomorphism is C^0- or C^∞-conjugated to the time 1 mapping of the flow of a vector field) has at least a twofold advantage.

Firstly the study of the diffeomorphism is reduced to the study of a vector field which in most cases reveals to be an easier task.

Secondly up to a homeomorphism the orbits of the vector field are kept invariant under the diffeomorphism, so that we find an invariant singular foliation (C^0 or C^∞) restricting the topological complexity of the diffeomorphism in essentially the same way as a first integral does.

A perhaps more important aspect can be seen in the study of periodic solutions for periodic time-dependent differential equations.

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In this context we would like to refer to the Floquet-Liapunov theory for a linear periodic system of differential equations stating among other things that the system can be transformed into an autonomous linear system by means of a coordinate change given by a periodic matrix function.

Let us now take X to be a more general T-periodic system of differential equations on $\mathbb{R}^n$ which we want to study in the neighbourhood of some T-periodic solution γ. As usual we associate to X an autonomous system of differential equations or vector field $Y = X + \frac{\partial}{\partial t}$ defined on $\mathbb{R}^{n+1} = \mathbb{R}^n \times \mathbb{R}$.

Because of the T-periodicity of X we can consider **Y** to be a vector field in $\mathbb{R}^n \times S$ where $S = \mathbb{R}/_{T\mathbb{Z}}$. For simplicity in exposition let us suppose that γ is the zero solution, i.e. $\Upsilon = \{0\} \times S$.

We take $f : \mathbb{R}^n \times \{0\} \circlearrowleft$ to be the first return mapping (Poincaré mapping) associated to Y, which in this case is $x \to \gamma_X(x,T)$ where γ_X denotes the global solution of X.

In analogy with the Floquet-Liapunov theory we can state that f C^r-embeds in a flow if and only if there exists a C^r diffeomorphism $H : \mathbb{R}^n \times S \circlearrowleft$, $(x,t) \to (H_t(x),t)$ with the property that this coordinate change H transforms the vector field Y into an expression $Z + \frac{\partial}{\partial t}$ with Z autonomous.

In that way the study of the diffeomorphism or the study of a system of differential equations to which can be associated a diffeomorphism (in casu the Poincaré mapping) is then reduced to a further investigation of an autonomous vector field in a space of the same dimension.

In this lecture note we deal with germs of diffeomorphisms f in the plane satisfying a so called Łojasiewicz inequality, exhibiting a characteristic line and having a 1-jet which can be expressed as $R+N$ with N nilpotent and $R^p = I$ for some $p \in \mathbb{N}$.

For exact definitions we refer to the first chapter. Roughly spoken the first condition means that the diffeomorphism is not too degenerate, although the condition is rather weak since all germs of diffeomorphisms showing up in generic n-parameter-families of diffeomorphisms, for whatsoever n, are of Łojasiewicz-type. The second condition is one of good sense; as a matter of fact in the other case the orbits indefinitely spiral around the fixed point and the study of this phenomenum is already fairly complicated and not completely understood in the vector field case.

The third condition means that we do not pay attention, except in the introductory remarks in chapter I, to the already well known diffeomorphisms like the hyperbolic and partially hyperbolic ones, as well as to diffeomorphisms whose associated R (semi simple part of the 1-jet) is an irrational rotation.

In all the cases treated here we find for the diffeomorphism a same kind of decomposition in parabolic, elliptic and hyperbolic sectors as for an R-equivariant vector field X. This X has the property that up to a C^∞ change of coordinates the ∞-jet of f is the same as the ∞-jet of $R \circ X_1$ where X_1 is the time 1-mapping of the flow X_t of X. Moreover the union of the boundaries of these sectors is a C^∞ image of the union of the boundaries for the X-decomposition.

Let us remark that in case R = Identity these sectors for f are "invariant" sectors while for general R we have for each sector S that f leaves "invariant" $\bigcup_{i=0}^{p} f^i(S)$ with $f^p(S)$ " = "S.

Knowing that in the Łojasiewicz type-case vector fields only have but one topological model of attracting, expanding, hyperbolic and elliptic sector (up to C° conjugacy) we in this work prove the same for the diffeomorphism (take the case R = Id), except for the hyperbolic sector.

We however show that in the interior of a hyperbolic sector orbits only stay a finite number of iterates. We use all this to prove that the diffeomorphism f (case R = Id) is weakly-C°-conjugated to the time 1-mapping X_1. Such f as we deal with is hence weakly-C°-embeddable in a flow and is up to weak-C°-conjugacy determined by some finite jet.

These results can be ameliorated if we do not allow certain partially hyperbolic singularities in a desingularisation of X obtained after successive blowing up.

Then as a matter of fact we find that f is C^{∞}-conjugated to X_1 on the union of parabolic and hyperbolic sectors.

Hence under the just mentioned extra assumption (which we only need inside the hyperbolic sectors) f is C°-conjugated to X_1 and is up to C°-conjugacy determined by some finite jet.

The elliptic sectors give C^{∞} problems, even under these extra conditions on the desingularisation. Under these extra conditions we are able to describe a complete (infinite dimensional) C^{∞} modulus for flat C^{∞} conjugacy (conjugacy by means of C^{∞} diffeomorphisms which are infinitely near the identity).

The reason essentially is that a flat C^∞ conjugacy between two elliptic sectors is uniquely determined in a conic neighbourhood of each of the two boundary lines. These uniquely defined diffeomorphisms do not need to match together in the middle of the sector and this obstruction can be fully described.

At least for a large class of germs of diffeomorphisms in $\mathbb{R}^2$ we so prove that the whole C^∞ structure only depends on the ∞-jet.
In other cases we get that this definitely is not the case. In many cases we show the diffeomorphism to be C^0 determined by some finite jet so that the investigation of the topological structure of the diffeomorphism becomes a problem concerning polynomial vector fields.

In order to make the lecture note accessible for non-specialists we added an extensive introduction in chapter I.
It contains besides the definition of most notions, a list of well known facts related to our study and a description of the main technique, namely the blowing-up method.
Moreover in chapter I we enumerate all our results in a rather self-contained way with a guide for travelling through the proofs; at the end we present some nice applications.
The rest of this note is then completely devoted to the proof of the theorems.

Some of the results in this paper have first been announced and proved in limited cases by Rodrigues*and Roussarie during a stay of the first at the university of Dijon.
The method of proof has been adapted and completed by Dumortier and Roussarie during a sejourn of both authors at the "Institut des Hautes Etudes Scientifiques" in Bures-s-Yvette.
The writing has essentially been finished while Dumortier remained at the university of Dijon.
We want to thank the mentioned institutions for their hospitality.

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Chapter I : Introduction, definitions, formal study and statement of the results

In §1 we introduce the problem, sketch some well known (and sometimes less known) related results and we situate the problem in its natural environment.

§2 contains a short description of the blowing up method as well as some results concerning singularities of vector fields of Łojasiewicz type in $\mathbb{R}^2$. We end this paragraph by giving a finite list of types of singularities to whose study our problem can be reduced by means of our fundamental theorem. We state this theorem in §3.

In §4 we discuss the decomposition in sectors for a singularity of a vector field of Łojasiewicz type with characteristic orbits and we say some words about characteristic lines.

In §5 we state the results concerning associated characteristic lines and decompositions in sectors for diffeomorphisms of Łojasiewicz type. We reduce the statements to our fundamental theorem.

§6 contains the statements and a preliminary elaboration of our principal C^∞ results, while the same is done in §7 for the C° results.

In §8 we give some applications.

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§1. Introduction

Some definitions

We will use following definitions and notations :

Diff(n) : the space of C^{∞} germs of diffeomorphisms in $0 \in \mathbb{R}^n$ having the origin as a fixed point.

$\mathcal{E}(n)$: the ring of germs of C^{∞} functions in $0 \in \mathbb{R}^n$

V(n) : the space of germs of C^{∞} vector fields in $0 \in \mathbb{R}^n$ vanishing in 0.

The symbol $\sim$ resp. $\sim_k$ placed above an element of Diff(n), V(n), $\mathcal{E}$(n) means that we consider the ∞-jet, resp. the k-jet of that element. We sometimes also use $j_k(.)(0)$ or $j_{\infty}(.)(0)$.

The flow of $X \in V(n)$ will be denoted by X_t instead of the often used $\phi_{X,t}$.

Definition 1.1

Let $g_1, g_2 \in$ Diff(n), we say that g_1 and g_2 are C^r-conjugated ($r \in \{0\} \cup \mathbb{N} \cup \{\infty\}$) if there exist local representatives $\check{g}_1$ and $\check{g}_2$ of resp. g_1 and g_2 defined on the resp. neighbourhoods V_1 and V_2 of 0 and if there exists some C^r diffeomorphism $h : V_1 \to V_2$ such that $h^{-1} \circ \check{g}_2 \circ h(x) = \check{g}_1(x) \quad \forall\, x \in V_1$ as long as both sides are defined.

(C^0 diffeomorphism means homeomorphism and C^0-conjugacy is also called topological conjugacy).

Definition 1.2

Let X, Y $\in$ V(n), we say that X and Y are C^r-conjugated ($r \in \{o\} \cup \mathbb{N} \cup \{\infty\}$) if there exist local representatives $\check{X}$ and $\check{Y}$ of resp. X and Y defined on the resp. neighbourhoods V_1 and V_2 of 0 and if there exists some C^r diffeomorphism $h : V_1 \to V_2$ such that $h^{-1} \circ \check{Y}_t \circ h(x) = \check{X}_t \quad \forall x \in V_1$ and $\forall t \in \mathbb{R}$ as long as both sides are defined (if $r \in \mathbb{N} \cup \{\infty\}$ the last condition can be formulated as : $h_*(\check{X}) = \check{Y}$).

Definition 1.3

Let g $\in$ Diff(n). We say that g C^r-embeds in a flow ($r \in \{o\} \cup \mathbb{N} \cup \{\infty\}$) if there exists X $\in$ V(n) such that g is C^r-conjugated to X_1.

Definition 1.4

Let g $\in$ Diff(n). We say that g is C^r determined by its k-jet $\tilde{g}^k$ ($k \in \mathbb{N} \cup \{\infty\}$) if $\forall$ f $\in$ Diff(n) with $\tilde{f}^k = \tilde{g}^k$ we have that f is C^r-conjugated to g.

In that case we say that the k-jet $\tilde{g}^k$ is determining for C^r-conjugacy or C^r-determining. g is called finitely C^r-determined if some finite jet of g is C^r-determining.

Definition 1.5

Let X $\in$ V(n). We say that X is C^r determined by its k-jet $\tilde{X}^k$ ($k \in \mathbb{N} \cup \{\infty\}$) if $\forall$ Y $\in$ V(n) with $\tilde{Y}^k = \tilde{X}^k$ we have that Y is C^r-conjugated to X.

In that case we say that the k-jet $\tilde{X}^k$ is determining for C^r-conjugacy or C^r-determining.

X is called finitely C^r-determined if some finite jet of X is C^r-determining.

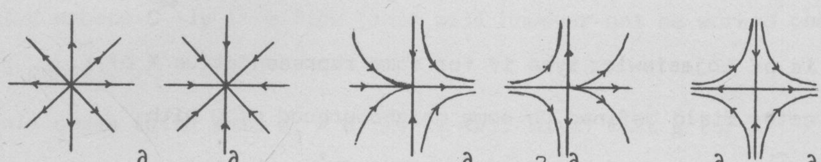
Situation of the problem and some related results

Let us first recall some well known results concerning the questions mentioned in the summary.

i) If $g \in \text{Diff}(n)$ is hyperbolic (this means that all the eigenvalues of $j_1(g)(0)$ lie off the unit circle, then by the theorem of Hartman [9] we know that g is topologically determined by its 1-jet in 0. By the theorem of Sternberg [19] we know that such hyperbolic g C^∞-ly embeds in a flow if and only if it formally embeds in a flow while it was already known by Lewis Jr. [11] that g or g^2 formally embeds in a flow (and hence C^∞-ly).

ii) For semi-hyperbolic diffeomorphisms in Diff(n) with 1 real eigenvalue having modulus 1 and all other eigenvalues with modulus $\neq 1$, we know that such a diffeomorphism always has a C^r center manifold (for any $r \in \mathbb{N}$) (A center - manifold W^c for g is an invariant manifold containing 0, such that the spectrum of $j_1(g|W^c)(0)$ lies on the unit circle and W^c has maximal dimension with respect to that property). In the case Diff(2) we can now look at $j_r(g|W^c)(0)$ for W^c being any C^r center manifold. These jets do not depend on the particular choice of C^r center manifold (see [22]) and except for a set of ∞-codimension (a set which is hence avoidable by generic C^∞ m-parameter families of diffeomorphisms) we may assume that one of these jets is different from the identity. In chapter III of this paper we prove that in that case there must exist (a not necessarily unique) C^∞ center manifold. We do not believe this result to be original although we never found it in the literature.

It is also known ([11] that in that case g or g^2 formally embeds in a flow. In chapter III we obtain that g or g^2 C^∞-ly embeds in a flow. Concerning C^r results with respect to these and other semi-hyperbolic points we refer to Takens [20]. On the other hand it is a well known result that the corresponding vector fields are C° determined by their r-jet where r is the lowest number such that $j_r(X|W^c)(0) \neq 0$; the different topological types may be classified by the 5 models :



$\pm (x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y})$ $\qquad$ $\pm y \frac{\partial}{\partial y} + x^2 \frac{\partial}{\partial x}$ $\qquad$ $x \frac{\partial}{\partial x} - y \frac{\partial}{\partial y}$

iii) In the sequel we will now exclusively consider germs $g \in \mathrm{Diff}(2)$ whose 1-jet in 0 has all its eigenvalues on the unit circle.

By the Jordan normal form theorem we may write $j_1(g)(0) = R+N$ with R the semi-simple part and N the nilpotent part. Following theorem of Takens [23] is crucial for further elaboration in the paper.

<u>Theorem 1.1</u>

If $\tilde{g} \in \widetilde{\mathrm{Diff}(2)}$ (remember : ~ stands for ∞-jet) and R is the semi-simple part of $\tilde{g}^1$ (= $j_1(\tilde{g})(0)$) with eigenvalues $e^{\pm 2\pi i \alpha}$ or ± 1 then there is a unique $\tilde{X} \in \widetilde{V(2)}$ invariant under R such that up to a C^∞ change of coordinates $\tilde{g}$ is equal to $R \circ \tilde{X}_1$ where $\tilde{X}_t$ denotes the formal flow associated to $\tilde{X}$.

Moreover, $\forall\, k \in \mathbb{N} \cup \{\infty\}$ the k-jet of $\tilde{X}$ only depends on the k-jet of $\tilde{g}$.

We will call $R \circ \tilde{X}_1$ the formal normal form of $\tilde{g}$.

We see that $\tilde{X}^1 = R^{-1}N$ which is always zero except if $R = \pm I$ in which case $\tilde{X}^1 = \pm N$ (or even $= N$ up to a linear change of coordinates).

Definition 1.6

Let $g \in \mathrm{Diff}(2)$. We say that g is of Łojasiewicz type (resp. algebraically isolated) if g has a formal normal form $R \circ \tilde{X}_1$ such that $\tilde{X}$ is of Łojasiewicz type (resp. is algebraically isolated).

We recall:

$\tilde{X} \in \widetilde{V(n)}$ is of Łojasiewicz type if for some representative X of $\tilde{X}$ (X a C^∞ vector field defined on some neighbourhood of 0 with $j_\infty(X)(0) = \tilde{X}$) we may find constants k, c, δ in $\mathbb{R}^+_o$ such that $\|X(x)\| \geq c\|x\|^k$ $\forall\, x$ with $\|x\| < \delta$ where $\|.\|$ denotes the euclidean norm on $\mathbb{R}^n$.

$\tilde{X} \in \widetilde{V(n)}$ is algebraically isolated if the ideal in $\widetilde{\mathcal{E}(n)}$ generated by the component functions of $\tilde{X}$ contains some power of the maximal ideal.

The condition of being of Łojasiewicz-type is less restrictive than being algebraically isolated.

For more details see [6].

Suppose now that $\tilde{g}^1$ has a semi-simple part R with eigenvalues $e^{\pm 2\pi i\alpha}$ and α irrational. Then the $\tilde{X}$ in the formal normal form has a 1-jet $\tilde{X}^1 = 0$, and since $\tilde{X}$ must be R-invariant - hence invariant under all rotations around the origin - we know that up to a C^∞ coordinate change $\tilde{X}$ can be written in polar coordinates as

$$\left(\sum_{i=1}^{\infty} a_i r^{2i}\right) \frac{\partial}{\partial\theta} + \left(\sum_{j=1}^{\infty} b_j r^{2j}\right) r \frac{\partial}{\partial r}$$

If we suppose g to be of Łojasiewicz type then there is some a_i or some b_j which is not zero. Moreover up to a set of ∞-codimension we even will have an expression

$$\left(\sum_{i=i_0}^{\infty} a_i r^{2i}\right)\frac{\partial}{\partial\theta} + \left(\sum_{j=j_0}^{\infty} b_j r^{2j}\right) r\frac{\partial}{\partial r}$$

with $a_{i_0} . b_{j_0} \neq 0$ for some i_0 and $j_0 \in \mathbb{N}$

In case $j_0 \leqslant i_0$ the techniques developed in this paper permit to prove that g embeds C^{∞}-ly in a flow (this will however not be worked out in this paper).
In all cases (with some $b_j \neq 0$) it is well known that g (or g^{-1} - depending on the sign of b_{j_0}) is topologically conjugated to the standard contraction $(x,y) \rightarrow (\frac{1}{2}x, \frac{1}{2}y)$.

iv) Finally we come to the case where $R^n = I$ for some $n \in \mathbb{N}$ and we moreover suppose that for the associated normal form the formal vector field $\tilde{X}$ is of Łojasiewicz type and has a characteristic orbit.

Definition 1.7 (characteristic orbit)

A vector field X on $\mathbb{R}^n$ with $X(0) = 0$ has a characteristic orbit in 0 if for some neighbourhood V of 0 there exists an integral curve $t \rightarrow X_t(y_0)$ remaining in V for $t \geqslant 0$ (resp. $t \leqslant 0$) and such that

- $\| X_t(y_0) \| > 0 \quad \forall t \geqslant 0$ (resp. $t \leqslant 0$)
- $X_t(y_0) \rightarrow 0$ for $t \rightarrow \infty$ (resp. $t \rightarrow -\infty$)
- the function $t \rightarrow \dfrac{X_t(y_0)}{\|X_t(y_0)\|}$ from $\mathbb{R}^+$ (resp. $\mathbb{R}^-$) to S^{n-1} tends to a limit when $t \rightarrow +\infty$ (resp. $t \rightarrow -\infty$)

Let us in this case call $\gamma = \{X_t(y_0) \mid t \in [0,\infty[\} \cup \{0\}$ (resp. $\gamma = \{X_t(y_0) \mid t \in]-\infty,0]\} \cup \{0\}$) a characteristic line for X.

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