Lecture Notes in Mathematics

Edited by A. Dold and B. Eckmann

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Geometries and Groups

Proceedings, Berlin 1981

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Proceedings of a Colloquium Held at the
Freie Universität Berlin, May 1981

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Dedicated to

Professor Dr. Hanfried Lenz

on the occasion of his 65th birthday

This volume contains the proceedings of a colloquium in honour of Prof. Dr. Hanfried Lenz, organized by the Freie Universität Berlin in May 1981 to mark both Prof. Lenz's 65th birthday and his retirement (from formal duties, not from mathematics !). Though Prof. Lenz worked in many areas of mathematics, his main interest has been in geometry and - for about the last 10 years - in particular in finite geometries and designs (for more details on his work see the following "Geleitwort"). It was decided to focus attention on the combinatorial and group theoretic aspects of geometry. Five survey lectures (of two hours each) were invited (given by F. Buekenhout, J. Doyen, D.R. Hughes, U. Ott and K. Strambach); the corresponding papers constitute the first part of this volume. There were also about 30 contributed talks (which were not restricted to the area mentioned above); 11 contributed papers fitting into this area form the second part of this volume which will hopefully be of interest to anyone working in geometries and groups.

We finally have one more remark: it was a great pleasure to be able to present Prof. Lenz with an ingenious machine (designed and built by Th. Beth and W. Fumy of the Universität Erlangen) which visibly displays some of the applications of combinatorial theory. The article by Fumy deals with the mathematical background and the powers of this delightful device.

Berlin and Gießen, July 1981

M. Aigner

D. Jungnickel

GELEITWORT

Dieser Band ist Hanfried Lenz zu seinem 65. Geburtstag gewidmet. Daher wollen wir uns hier sein bisheriges mathematisches Werk vor Augen führen, was bei der großen Zahl seiner Veröffentlichungen allerdings nur in Auswahl geschehen kann. Ein solches Auswählen ist schwer, da Hanfried Lenz auf vielen verschiedenen Gebieten der Mathematik gearbeitet hat und ein Abwägen der Bedeutung immer subjektiv sein wird.

Am Anfang seines mathematischen Schaffens stehen einige kleinere Arbeiten zur Analysis und Funktionentheorie (1951/2), und auf diesen Problemkreis kommt er später (1956/7) noch einmal zurück. Aber schon 1952 wendet sich sein Interesse der Geometrie zu und zwar zuerst der Theorie der projektiven Ebenen, die eigentlich erst seit den vierziger Jahren ein Eigenleben gewann. 1953 entdeckt er das erste Beispiel einer endlichen projektiven Ebene , in der einige, aber nicht alle vollständige Vierecke kollineare Diagonalpunkte haben[1]. 1954 erscheint seine Arbeit "Kleiner Desarguesscher Satz und Dualität in projektiven Ebenen"[2], in der er die projektiven Ebenen nach der Menge der Paare (P,g) (P Punkt auf Gerade g) klassifiziert, für welche die Gruppe der (P,g)-Kollineationen (alle Geraden durch P und alle Punkte auf g bleiben fest) transitiv auf den Punkten $\neq$P einer Geraden $\neq$g durch P wirkt. Diese Klasseneinteilung wurde 1957 durch Barlotti verfeinert, indem er auch die nichtinzidenten Paare (P,g) einbezog. Als Lenz-Barlotti-Klassifikation bezeichnet gehört diese Einteilung seitdem zum Standardwerkzeug in der Theorie der projektiven Ebenen. Ebenfalls 1954 erscheint eine umfangreiche Arbeit "Zur Begründung der analytischen Geometrie"; hier (und in einer im gleichen Band der Sitzungsbericht der Bayerischen Akademie der Wissenschaften erschienen kleineren Note) wird die projektive Geometrie beliebiger (auch unendlicher) Dimension ≥ 3 aus Verknüpfungsaxiomen entwickelt und ebenso die affine Geometrie unter Benutzung eines Parallelismus in der Geradenmenge. Ebenso geht Hanfried Lenz 1958 in seinem mathematikdidaktisch orientierten Beitrag "Ein kurzer Weg zur analytischen Geometrie"[2] vor. Hier weist er auf die von Artin, Baer, Dieudonné verwendete koordinatenfreie Schreibweise hin ("an Einfachheit und Allgemeinheit der älteren formalen Behandlung der linearen Algebra überlegen") und bemerkt in einer

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1) Arch.Math. <u>4</u>, 327-330
2) Jahresber. DMV <u>57</u>, 20-31
3) Math.Phys.Semesterber. <u>6</u>, 57-67

Fußnote (hier als in ihrer Formulierung für den Verfasser typisch zitiert): "Die unter Fußnote 6 zitierten Arbeiten Lenz 1953/4 geben in dieser Hinsicht Beispiele, wie man es nicht machen soll". Auch seine 1961 bzw. 1965 erschienenen Bücher "Grundlagen der Elementarmathematik" (3.Aufl. 1975) und "Vorlesungen über projektive Geometrie" gehen teilweise auf diese Beschäftigung mit den Grundlagen der projektiven und affinen Geometrie zurück. Beide Werke enthalten viel bemerkenswert Neues, u.a. über quadratische Formen. Im zweiten dieser Bücher heißt es über die Bedeutung der projektiven Geometrie: "Wenn man von der z.Zt. blühenden Erforschung der projektiven Ebenen absieht, bietet die projektive Geometrie heute nicht noch viele offene Fragen: Sie ist aber unentbehrlich 1. zur Zusammenfassung vieler klassischer geometrischer Theorien im Sinne des Erlanger Programms von Felix Klein, 2. für die Begründung der nichteuklidischen Geometrie und 3. als Vorstufe für das Studium des sehr ausgedehnten und schwierigen Fachgebietes der algebraischen Geometrie. Für diese Zwecke erkennt auch Dieudonné die Existenzberechtigung der projektiven Geometrie an, die er im übrigen veraltet nennt. Da er selbst schöne Beiträge zur projektiven Geometrie geliefert hat, sollte man m.E. diese Kritik nicht tragisch nehmen, sondern vielmehr die vielen bewährten Schlußweisen und Ergebnisse der projektiven Geometrie in den modernen Aufbau der Mathematik einbauen."

Aus den Jahren 58, 59 sind drei Arbeiten[4] hervorzuheben, die Beiträge zum Helmholtzschen Raumproblem in endlichdimensionalen reellen bzw. komplexen Vektorräumen liefern. Dem Bemühen, den mathematischen Hintergrund des im Mathematikunterricht der Schule Gelehrten aufzuklären, das Hanfried Lenz schon zum Schreiben seiner "Grundlagen der Elementarmathematik" veranlaßt hatte, entspringt 1967 eine die ordnungstheoretischen Grundlagen aufklärende Begründung der Winkelmessung[5]. Seine Stellung zu Fragen des Mathematik- und insbesondere des Geometrieunterrichts findet man eindringlich beschrieben in dem Beitrag "A bas Euclide - vive Bourbaki?" aus dem Jahr 1963:[6] Man sollte "Euklid" durch "Bourbaki" erneuern und ergänzen, aber nicht ersetzen (leider wurde diese Mahnung in den folgenden Jahren zu wenig berücksichtigt); eine besonders bemerkenswerte von drei Thesen lautet: "Psychologie und Didaktik haben den Vorrang vor logischer Abstraktion und Axiomatik, sind aber damit verträglich."

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4) Arch.Math. 8,477-480, Math.Ann.135,244-250, u. 137,150-166.
5) Math. Nachr. 33, 363-375
6) Praxis d.Math. 5, Heft 4, 85-87

1975/6 wendet sich Hanfried Lenz einem neuen Forschungsgebiet zu, der endlichen Geometrie: Zwei gemeinsam mit D. A. Drake geschriebene Arbeiten[7] bringen einen Durchbruch in der Existenzfrage für endliche Hjelmslev-Ebenen; 1977 beschreibt er[8] ein Konstruktionsverfahren für spezielle endliche Inzidenzstrukturen (mittels Einsetzen von Inzidenzmatrizen in Inzidenzmatrizen), das sich in der Folge in Arbeiten von Lenz und anderen Verfassern als sehr fruchtbar in der Theorie der Blockpläne, Steiner-Systeme und anderer Design-Klassen erweist. Gemeinsam mit Th. Beth und D. Jungnickel schreibt Hanfried Lenz an einem umfangreichen Buch "Design Theory" , das viele neue Ergebnisse von ihm und seinen Mitarbeitern enthalten wird. Für dieses Buch und für seine weiteren Forschungen auf dem Gebiet der "Designs" wünschen wir Hanfried Lenz auch in unserem Interesse viel Erfolg.

Im Namen aller Freunde, Mitarbeiter und Kollegen von Hanfried Lenz

Günter Pickert

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7) Abh.Math.Sem. Hamburg 44, 70-83; Bull.Amer.Math.Soc. 82, 265-267

8) In: Beiträge zur Geometrischen Algebra; 225-235

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THE BASIC DIAGRAM OF A GEOMETRY

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1. Introduction.

The language of diagrams for geometries has been devised by Tits in the 1950's and it has been developed by him toward a complete combinatorial-geometric theory of the simple groups of Lie-Chevalley type [27]. Recently the author observed that the basic ideas of the diagram language could be applied to geometries arising from some of the sporadic groups [1]. In this context inspiration came primarily from the deep investigation of groups with a class of 3-transpositions due to B.Fischer [8]. This discovery of geometries for the sporadic groups gave a new impulse to some of the earliest ideas of Tits. For an excellent historical introduction to these ideas we refer to Tits [29].

During the last few years, an important activity on diagram geometries has been growing and it seems likely that this trend is going to increase in the near future.

There are several reasons for this :

a) diagram geometry provides a unified setting for many geometric theories in a field where this is badly needed ;

b) as a consequence, known theories and known methods are better understood ;

c) as another consequence, an enormous potential of new problems and of variations on known problems or known results is easily made available ;

d) relationships with other fields are made easier.

The last statement is particularly well illustrated by the following examples :

- geometric interpretation of the finite simple groups and possible influence on the "revision" of the classification theory for these groups (in the sense of Gorenstein)
- representation theory in the spirit of the Feit-Higman theorem which is the subject of Ott's report at this conference (see also Ott [16]).
- the obvious connection with combinatorial topology (see Ronan [20], [21], Tits [30]).

In view of these characteristics it is desirable as always to work on firm foundations. This is the purpose of our paper. In [1],[3] basic concepts and general theorems are exposed. Various persons have pointed out shortcommings of different kinds in these papers and a series of suggestions and improvements have been made. Altogether the evolution of foundations with respect to [1] is sufficient to justify a completely new synthesis.

The author would like to acknowledge the help of G.Glauberman and M.Perkel for their most careful reading of [1] and their numerous suggestions **and A. Brouwer who made several corrections in this paper.**

2. *Geometries over a set*.

Let Δ be a set which can be seen as a set of indexes, a set of names such as point, line, plane,...,a set of colors, etc. Here elements and subsets of Δ will be called *types*.

A *geometry* Γ *over* Δ is a triple $\Gamma=(S,I,t)$ where S is a set (the *elements* of Γ), I is a symmetric and reflexive relation defined on S (the *incidence relation* of Γ) and t is a mapping of S onto Δ (the *type function* of Γ) such that (TP) the restriction of t to every maximal set of pairwise incident elements is a bijection onto Δ (*transversality property*).

Comments. 1) We shall also use the notation $\Gamma_i=t^{-1}(i)$ for each $i \in \Delta$. In other words, Γ_i is the set of all elements of Γ of type i.

2) In [1] we would rather speak of incidence structures and varieties instead of geometries and elements. The present vocabulary has been adopted in most recent talks on the subjet, in particular [29].

3) Notice that a geometry can be seen as a multipartite graph with distinct colors on the distinct components.

4) One could equivalently see a geometry over Δ as a set endowed with a partition indexed by Δ or see it as a quadruple (S,I,Δ,t) etc.

5) The axioms numbered (1), (2), (3) in [1] and required there from an incidence structure will not be required from a general geometry with the exception of (TP). There is indeed evidence that this more general viewpoint may be technically useful as is pointed out by Percsy [18], [19].

6) Strangely enough the concept of geometry as presented here appears very clearly in a paper of E.H.Moore as early as 1896 [14].

We shall need some more definitions. Let $\Gamma=(S,I,t)$ be a geometry over Δ.

A *flag* F of Γ is a (possibly empty) set of pairwise incident elements of Γ. Two flags F_1, F_2 of Γ are called incident and we write $F_1 \; I \; F_2$ if $F_1 \cup F_2$ is still a flag. Maximal flags are called chambers. The set of all chambers of Γ is denoted by Cham Γ.

The type of a flag F is the set t(F). If A is a subset of Δ then a flag of type A will also be called an A-flag. If A is reduced to a single element of Δ then an A-flag is also called an A-element. Hence if $i \in \Delta$, then Γ_i is the set of all i-elements of Γ or the set of all elements of type i. The cotype $t^*(F)$ of a flag F is the set Δ-t(F).

The residue of a flag F in Γ is the geometry $\Gamma_F=(S_F, I_F, t_F)$ over $t(S_F)=\Delta-t(F)$ defined by : S_F is the set of all elements of Γ not in F, incident with all elements in F, I_F (resp.t_F) is the restriction of I (resp.t) to S_F. Clearly axiom (TP) holds.

2.1.Proposition. If $\Gamma=(S,I,t)$ is a geometry over Δ and F is a flag of Γ then the residue Γ_F is a geometry over Δ-t(F).

The rank $r(\Gamma)$ of a geometry Γ over Δ is the cardinality of Δ i.e. the number of distinct types in which the elements of Γ fall. If F is a flag then the rank r(F) of F is the cardinality of F and the corank of F is the rank of Γ_F.

Two chambers are called adjacent if their intersection is a flag of corank 1. This provides clearly a graph structure on Cham Γ.
Let $\Gamma=(S,I,t)$ be a geometry over Δ and $\Gamma'=(S',I',t')$ be a geometry over Δ' such that $S' \subseteq S$, $I' \subseteq I$, $\Delta' \subseteq \Delta$ and t' is the restriction of t to S'. Then Γ' is a subgeometry of Γ. If in addition I' is the restriction of I to S' then Γ' is an induced subgeometry of Γ.

If Δ is a set then the unit geometry 1_Δ over Δ is the geometry $\Gamma=(\Delta,I,t)$ where I consists of all pairs $(x,y) \in \Delta^2$ and t is the identity map.

Comments 7) A geometry Γ determines a flag complex $F(\Gamma)$ in the sense of Tits [27] whose elements are the flags of Γ and in which the order relation is inclusion. The notion of flag complex is however more general than that of geometry since any graph and not just a multipartite graph determines a flag complex.

8) In the context of flag complexes the residue of a flag F as defined here, corresponds to the star of A or St A [27] which is also called the link of A in topology.

9) In [30], Tits introduces another approach of geometries based on chambers which was first suggested by L.Puig. A chamber system over the set Δ consists of a set C whose elements are called chambers together with a system of partitions of C indexed by Δ.
Clearly every geometry Γ determines a chamber system which we shall

denote abusively by Cham Γ: for every $i \in \Delta$, two chambers are called i-equivalent if their intersection is of cotype i; the equivalence classes of this relation determine a partition of Cham Γ corresponding to i.

Conversely every chamber system C determines a geometry $\Gamma(C)$ (see [30]). In general Γ does not coincide with Γ(Cham Γ); however this holds under fairly mild conditions [30] namely that Δ is finite and that Γ is strongly connected (see section 5).

10) A geometry of rank 0 consists of the empty set and the empty incidence relation.

Geometries of rank 1 consist trivially of any set and a uniquely determined incidence relation.

3. Morphisms.

Let $\Gamma=(S,I,t)$ and $\Gamma'=(S',I',t')$ be geometries over sets Δ and Δ' respectively.

A morphism from Γ to Γ' is a pair (α,β) where $\alpha:S \to S'$, $\beta:\Delta \to \Delta'$ are mappings such that $x\ I\ y$ implies $\alpha(x)\ I'\ \alpha(y)$ and $t' o \alpha = \beta o t$.

Equivalently a morphism is a mapping $\alpha:S \to S'$ such that α preserves incidence and type equality i.e. $t(x)=t(y)$ implies $t'(\alpha(x))=t'(\alpha(y))$.

A Δ-morphism is a morphism (α,β) such that β is the identity i.e. a morphism preserving types.

Isomorphisms, automorphisms, Δ-isomorphisms and Δ-automorphisms are defined in the standard usual way.

Comments 1) Important automorphisms of geometries which are not Δ-automorphisms are provided by dualities, polarities and trialities (see for instance [26]). Morphisms which are not isomorphisms tend to play an increasing role in geometry. This is the case of foldings in the theory of buildings [27] (see also [9], [31]).

Embedding a geometry into another geometry involves a morphism of course (see Percsy [18]).

The local theory of buildings developed by Tits [30] and the theory of universal covers of chamber systems as treated by Ronan [20] rely on a special kind of morphism or local isomorphism.

2) For every geometry Γ over Δ there is a canonical Δ-morphism on the unit geometry 1_Δ over Δ namely the type function t.

3) Under a morphism α the image of a flag F is a flag $\alpha(F)$ and the image $\alpha(\Gamma_F)$ is contained into $\alpha(\Gamma)_{\alpha(F)}$.

4) Here is a fundamental procedure leading to interesting subgeometries in various contexts. Let α be a morphism of Γ into Γ (endomorphism). Then the induced subgeometry Abs α(absolute of α) consists of all elements X of Γ such that $\alpha(X)$ I X . If α is a Δ-morphism then Abs α consists of all elements fixed by α. If α is a polarity of a projective space then Abs α is a polar space or better Abs α determines a covering of a polar space (see below). If α is a triality of a polar space of type D_4 then Abs α determines a covering of a generalized hexagon.

We shall have some use of the following definition given by Tits in [30]. A Δ-morphism α is a covering if α is surjective and if for every $x \in S$ the restriction of α to the residue Γ_x is an isomorphism onto $\alpha(\Gamma)_{\alpha(x)}$. If there is such a covering then we shall also say that Γ is a covering of Γ'.

4. Thickness.

A geometry Γ is firm (resp. thick) if every non-maximal flag F of Γ is contained in at least two (resp. three) chambers of Γ.
A geometry Γ is thin if every flag of corank 1 is contained in exactly two chambers of Γ.

4.1. Proposition. A geometry Γ is firm (resp. thick, thin) if and only if every rank 2 residue of (a flag in) Γ is firm (resp. thick, thin).
Proof. Straightforward.

Comments 1) Things, polyhedra and polytopes provide examples of thin geometries.
2) A geometry Γ over Δ is a covering of the unit geometry 1_Δ if and only if every non-empty flag of Γ is contained in exactly one chamber.

Let $\Gamma=(S,I,t)$ be a geometry over Δ and let $i \in \Delta$. Then the i-order of Γ is the set of all numbers n-1 where n is the number of chambers containing some flag of cotype i. If Γ_F is some residue of Γ with $i \in t(\Gamma_F)$ then the i-order of Γ_F is contained in the i-order of Γ.

A thin geometry is characterized by the property that all of its i-orders are equal to 1. A covering of the unit geometry is characterized by the property that all of its i-orders are equal to 0.

If the i-order of Γ is reduced to a single number for each i then Γ is called order regular. In that case every residue Γ_F is also order regular.

5. Connectedness.

We shall say that a geometry $\Gamma=(S,I,t)$ over Δ is strongly connected if for every distinct i,j in Δ, $t^{-1}(i) \cup t^{-1}(j)$ is a connected graph for the incidence relation and if the same property holds in every residue Γ_F where F is a flag of Γ.

Comments 1) This is axiom (2) in [1]. The terminology is that of Tits [29]. In [30], Tits uses a weaker notion of connectedness and a notion of residual connectedness which is equivalent to strong connectedness.
2) Notice that the empty geometry and any rank 1 geometry are strongly connected. From the point of view of graph theory or topology this is a good reason to distinguish strong connectedness and connectedness.
3) In [30] a geometry $\Gamma=(S,I,t)$ is called connected if the graph (S,I) is connected (which implies S non empty).

Every strongly connected geometry of rank $\geqslant 2$ is connected. Γ is called simply connected (resp. strongly simply connected) if it is connected (resp. strongly connected) and if every covering by a connected geometry is an isomorphism (resp. if all residues of flags of corank $\geqslant 3$ in Γ are simply connected).
4) Observe that strong connectedness does not imply firmness : the unit geometry over Δ provides a counter-example.
5) If Γ is strongly connected then every residue Γ_F is strongly connected.
6) Assume Γ is a firm geometry of finite rank. Then Γ is strongly connected if and only if the set Cham Γ provided with the adjacency relation is a connected graph and the same property holds for Cham Γ_F where F is any flag of Γ.
The proof of this property is fairly straightforward. It has not appeared fully in print. For a partial proof see A.Valette [31].

6. Direct sums of geometries.

The concept of direct sum is fully recognized and used in Tits [25]. For a more explicit and detailed study we refer to A.Valette [31] whose work will be closely followed here.

Let J be a set of indices and let $(\Delta_j)_{j \in J}$ be a family of sets.

For each $j \in J$ let $\Gamma_j=(S_j,I_j,t_j)$ be a geometry over Δ_j. We assume that the Δ_j's are pairwise disjoint as well as the S_j's.

6.1. The direct sum of the geometries Γ_j is the geometry $\Gamma=\bigoplus_{j\in J}\Gamma_j=(S,I,t)$ defined as follows:

1) Δ is the union of the Δ_j's ;
2) S is the union of the S_j's ;
3) $I|S_j=I_j$ and $x\ I\ y$ whenever x and y belong to dinstinct components Γ_j,Γ_k respectively ;
4) $t|S_j=t_j$.

Examples 1) A direct sum of rank 1 geometries is a complete multipartite graph and conversely. A direct sum of two rank 1 geometries is called a generalized digon. These rank 2 geometries play a fundamental role in the theory.

2) If Γ is a geometry over Δ and 0 denotes the empty geometry over the empty set then Γ is obviously isomorphic to $\Gamma\oplus 0$.

3) A unit geometry of rank n is the direct sum of n unit geometries of rank 1.

Properties (see A.Valette [31]).

6.1. F is a flag (resp.chamber) of $\bigoplus_{j\in J}\Gamma_j$ if and only if $F\cap S_j$ is a flag (resp.chamber) of Γ_j for every $j\in J$.

6.2. F is a flag of corank one of $\bigoplus_{j\in J}\Gamma_j$ if and only if there is a unique $j\in J$ such that $F\cap S_j$ is a flag of corank one of Γ_j and for $k\neq j$, $F\cap S_k$ is a chamber of Γ_k.

6.3. There is a canonical bijection from Cham $\bigoplus_{j\in J}\Gamma_j$ onto the cartesian product $\prod_{j\in J}$ Cham Γ_j which completely describes the adjacency relation on Cham $\bigoplus_{j\in J}\Gamma_j$.

6.4. Let F be a flag of $\bigoplus_{j\in J}\Gamma_j$ and $\Gamma_{j(F\cap S_j)}$ be the residue of $F\cap S_j$ in Γ_j. Then Γ_F is isomorphic to the direct sum of the geometries $\Gamma_{j(F\cap S_j)}$.

6.5. $\bigoplus_{j\in J}\Gamma_j$ is firm (resp.thick, thin) if and only if each Γ_j is firm (resp.thick, thin).

6.6. $\bigoplus_{j\in J}\Gamma_j$ is strongly connected if and only if each Γ_j is strongly connected.

6.7. $Aut_\Delta(\bigoplus_{j\in J}\Gamma_j)$ (the group of all Δ-automorphisms) is isomorphic to the direct product $\prod_{j\in J} Aut_{\Delta_j}(\Gamma_j)$. Furthermore the first group is chamber-transitive if and only if each $Aut_{\Delta_j}(\Gamma_j)$ is chamber-transitive.

6.8. $\text{Rank} \bigoplus_{j\in J} \Gamma_j = \sum_{j\in J} \text{rank } \Gamma_j$.

We shall say that a direct sum is non-trivial if there is no empty component among the Γ_j and if there are at least two distinct components Γ_j.

As usual the usefulness of a direct sum concept is to give rise to conditions under which some given object decomposes into a direct sum of other (simpler) objects. We shall now report on a result of this kind which is the first non-trivial theorem in the theory and which plays a crucial role in all studies of diagram geometries.

7. The basic diagram of a geometry.

Let $\Gamma=(S,I,t)$ be a geometry over Δ.
We shall now introduce a graph structure on Δ, say $\Delta(\Gamma)$ induced by Γ, which we call the basic diagram of Γ because diagrams to be introduced later will appear as specializations of it.

A pair of distinct elements i,j of Δ are called joined, i.e. they constitute an edge of the basic diagram, whenever there is at least one flag F of cotype $\{i,j\}$ in Γ whose residue is not a generalized digon, i.e. whose residue is not a non-trivial direct sum of other geometries.

Comments 1) In [1] the basic diagram is used to introduce Theorem 2, up to the terminology.
Pasini [17] observed that the general theory developed in [3] does not require the full strength of the diagram concept as introduced in [1] but that it requires only the basic diagram. These ideas are made more explicit in the next sections.

7.1. Proposition. Let $\Gamma=(S,I,t)$ be a geometry over Δ and F a flag of cotype Δ' in Γ. Then the basie diagram $\Delta'(\Gamma_F)$ of the residue of F is a subgraph of the basic diagram $\Delta(\Gamma)$.

Proof. Straightforward.

Comments 2) In most well behaved geometries, $\Delta'(\Gamma_F)$ is actually an induced subgraph of $\Delta(\Gamma)$, i.e. two elements i,j of Δ' are joined in Δ' with respect to Γ_F if and only if they are joined in Δ with respect to Γ. Pasini [17] constructs examples in which $\Delta'(\Gamma_F)$ is not always