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Explicit Constructions of Automorphic L-Functions

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Explicit Constructions of Automorphic L-Functions

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Authors

Stephen Gelbart
Department of Theoretical Mathematics, The Weizmann Institute of Science
Rehovot 76100, Israel

Ilya Piatetski-Shapiro
Department of Mathematics, Tel-Aviv University
Ramat Aviv 69978, Israel
and
Department of Mathematics
Yale University
New Haven, CT 06520, USA

Stephen Rallis
Department of Mathematics, Ohio State University
Columbus, OH 43210, USA

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PREFACE

These previously unpublished manuscripts describe certain L-functions attached to automorphic representations of the classical groups.

Part A dates from 1983-84 and represents work of Piatetski-Shapiro and Rallis. The subject matter is a generalization of the method of Godement-Jacquet from $GL(n)$ to a simple classical group G.

Part B was written by Gelbart and Piatetski-Shapiro in the Fall of 1985, with an Appendix by all three authors. This work concerns a generalization of Rankin-Selberg convolution to $G \times GL(n)$, with G a classical reductive group of split rank n.

Parts A and B appear with their own Introductions and Bibliography. For a discussion of how these results are related to the recent works of F. Shahidi, the reader is referred to the 'Postscript' in the Introduction to Part B, and also to S. Gelbart and F. Shahidi's new paper on "Analytic Properties of Automorphic L-functions".

The expeditious preparation of the final form of this Lecture Note volume was done by Miriam Abraham of the Weizmann Institute, to whom we offer our thanks.

S. Gelbart

I. Piatetski-Shapiro

S. Rallis

January 1987

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CONTENTS

Chapter II : The Local Functional Equation

PART A : L-FUNCTIONS FOR THE CLASSICAL GROUPS

by

I. Piatetski-Shapiro and S. Rallis

Introduction.

These notes are based on lectures given by I.I. Piatetski-Shapiro at the Institute for Advanced Study in 1983-84. The notes were prepared by J. Cogdell, who made valuable remarks improving both the mathematics and the style.

The theory described here generalizes the method of Godement-Jacquet from $GL(n)$ to the simple classical groups of symplectic, orthogonal, and unitary type. In particular, it does not require the automorphic representations whose L-functions are analyzed to be generic. Even when $G = GL(n)$ (or more generally the unit group of a matrix algebra over a division algebra) it gives a new way of looking at the Godement-Jacquet zeta-function as a Rankin-Selberg type integral involving Eisenstein series on a much larger group. The basic identity relating this Rankin-Selberg integral to a global zeta-integral for $L(\pi, s)$ is established axiomatically in §1. These axioms are then treated separately for the various different classical groups in §§2-4, and the analytaic properties of the Eisenstein series are developed in §5.

§1. A formal identity.

In this section we will present an identity, which allows us to construct L-functions with Euler product associated to irreducible cuspidal automorphic representations of certain reductive groups. The construction is based on the classic Rankin-Selberg construction. It can be applied to all of the classical groups to yield new L-functions in certain cases and new integral representations for some previously known L-functions.

We take k to be a global field, $\mathbb{A}$ the adeles of k and I_k the ideles of k. Take G to be a reductive algebraic group with anisotropic center. (This means that if C is the center of G then $C_k/C_{\mathbb{A}}$ is compact.) Let H be another reductive algebraic group over k and assume we can embed $G \times G \overset{i}{\hookrightarrow} H$. Identify $G \times G$ with its image under i and let $G^d \subset H$ be the image of G under the composition of the diagonal embedding of $G \hookrightarrow G \times G$ and the embedding $i : G \times G \hookrightarrow H$.

Let P be a parabolic subgroup of H. Then we have an action of $G \times G$ on the flag variety $X = P \setminus H$ and X will decompose into orbits under the action of $G \times G$. A $G \times G$ orbit $X' \subset X$ will be called negligible if the stabilizer R' in $G \times G$ of a point $x' \in X'$ contains the unipotent radical N' of a proper parabolic subgroup of $G \times G$ as a normal subgroup. Let $x_0 \in X$ be the point of X corresponding to the coset $P1$ and X_0 its orbit. The stabilizer R_0 of x_0 in $G \times G$ is then just $P \cap (G \times G)$.

For our construction of L-functions we need that the following two conditions on the action of $G \times G$ on X be satisfied:

(1) The stabilizer R_0 of x_0 is G^d.

(2) If X' is any orbit other than X_0, then X' is negligible.

For this reason, we will refer to X_0 as the main orbit. Justification for the term negligible will be clear from our construction.

Now assume we have H, P, and $i : G \times G \hookrightarrow H$ such that conditions (1) and (2) are satisfied. Let $\delta : P \to k^x$ be the modulus character of P and let $\omega : I_k/k^x \to \mathbb{C}$ be any quasi-character such the $\omega \circ \delta$ is trivial on $G^d_{\mathbb{A}}$. Let $f(g;\omega) \in ind_{P_{\mathbb{A}}}^{H_{\mathbb{A}}}(\omega \circ \delta)$. (Our induction ind is not normalized, i.e., $f \in ind_{P_{\mathbb{A}}}^{H_{\mathbb{A}}}(\omega \circ \delta)$ iff $f(pg) = \omega(\delta(p))f(g)$ for $g \in H_{\mathbb{A}}$

and $p \in P_{\mathbb{A}}$.) Then to f we may associate the usual Eisenstein series

$$E_f(h;\omega) = \sum_{\gamma \in P_k \backslash H_k} f(\gamma h;\omega)$$

when this is absolutely convergent. If π is an irreducible cuspidal automorphic representation of G and $\tilde{\pi}$ its contragredient, then to $\phi_1 \in \pi$ and $\phi_2 \in \tilde{\pi}$ we may associate a Rankin-Selberg type L-function by setting

$$(1.1) \qquad L(\omega;\phi_1,\phi_2,f) = \int_{(G\times G)_k\backslash(G\times G)_{\mathbb{A}}} E_f((g_1,g_2);\omega)\phi_1(g_1)\phi_2(g_2)dg_1dg_2$$

Since ϕ_1 and ϕ_2 are cuspidal, this integral converges absolutely and inherits the analytic properties of the Eisenstein series $E_f(h;\omega)$.

A key property of these L-functions is that they will have an Euler product expansion. This will follow from the following Basic Identity.

Basic Identity:

$$\int_{(G\times G)_k\backslash(G\times G)_{\mathbb{A}}} E_f((g_1,g_2);\omega)\phi_1(g_1)\phi_2(g_2)dg_1dg_2 =$$

$$= \int_{G_{\mathbb{A}}} f((g,1);\omega) < \pi(g)\phi_1,\phi_2 > dg$$

where $< \phi_1,\phi_2 >$ is the bilinear Peterson inner product given by

$$< \phi_1,\phi_2 >= \int_{G_k\backslash G_{\mathbb{A}}} \phi_1(g)\phi_2(g)dg \ .$$

Proof:

We first insert the definition of $E_f(h,\omega)$ into the integral expression (1.1) for $L(\omega;\phi_1,\phi_2,f)$ and unfold.

$$L(\omega;\phi_1,\phi_2,f) = \int_{(G\times G)_k\backslash(G\times G)_{\mathbb{A}}} \left(\sum_{P_k\backslash H_k} f(\gamma(g_1,g_2))\right)\phi_1(g_1)\phi_2(g_2)dg_1dg_2$$

$$= \sum_{\gamma\in P_k\backslash H_k/(G\times G)_k} \int_{(G\times G)_k^\gamma\backslash(G\times G)_{\mathbb{A}}} f(\gamma(g_1,g_2)))\phi_1(g_1)\phi_2(g_2)dg_1dg_2$$

where $(G \times G)_k^\gamma = \{(g_1, g_2) \in (G \times G)_k | \gamma(g_1, g_2)\gamma^{-1} \in P_k\}$. Now, the double cosets $P_k \setminus H_k/(G \times G)_k$ parameterize the orbits of $(G \times G)_k$ on $X_k = P_k \setminus H_k$. We consider the main orbit and the negligible orbits separately.

a) Assume $\gamma_0 \in P_k \setminus H_k/(G \times G)_k$, $\gamma_0 = 1$, corresponding to the main orbit. Then $(G \times G)_k^{\gamma_0} = (G \times G)_k \cap P_k = G_k^d$. Then

$$I(\gamma_0) = \int_{(G\times G)_k^{\gamma_0} \setminus (G\times G)_{\mathbb{A}}} f(\gamma_0(g_1, g_2); \omega)\phi_1(g_1)\phi_2(g_2) dg_1 dg_2$$

$$\int_{G_k^d \setminus (G\times G)_{\mathbb{A}}} f((g_2, g_2)(g_2^{-1} g_1, 1); \omega)\phi_1(g_1)\phi_2(g_2) dg_1 dg_2 \ .$$

Since $\omega \circ \delta$ is trivial on $G_{\mathbb{A}}^d$ we have

$$f((g_2, g_2)(g_2^{-1} g_1, 1); \omega) = f((g_2^{-1} g_1, 1); \omega) \ .$$

If we now write $G \times G = G^d \;\; G_1$ where $G_1 = \{(g, 1) \in G \times G\}$ and write $(g_1, g_2) = (g_2, g_2)(g, 1)$ with $g = g_2^{-1} g_1$ then

$$I(\gamma_0) = \int_{G_{\mathbb{A}}} f((g,1); \omega) \left(\int_{G_k \setminus G_{\mathbb{A}}} \phi_1(g_2 g)\phi_2(g_2) dg_2 \right) dg$$

$$= \int_{G_{\mathbb{A}}} f((g,1); \omega) < \pi(g)\phi_1, \phi_2 > dg \ .$$

b) **Negligible orbits.**

Now let $\gamma \in P_k \setminus H_k/(G \times G)_k$ correspond to a negligible orbit. If we consider the action of $G \times G$ on $P \setminus H$, the stabilizer R^γ of $P\gamma$ in $G \times G$ is

$$R^\gamma = \{(g_1, g_2) | P\gamma(g_1, g_2) \in P\gamma\} = \{(g_1, g_2) | \gamma(g_1, g_2)\gamma^{-1} \in P\} = (G \times G)^\gamma \ .$$

By the assumption that the orbit $P\gamma(G \times G)$ is negligible, there is a proper parabolic $P^\gamma \subset G \times G$ whose unipotent radical N^γ is normal in R^γ. Then

$$I(\gamma) = \int_{R_k^\gamma \setminus (G\times G)_{\mathbb{A}}} f(\gamma(g_1, g_2); \omega)\phi_1(g_1)\phi_2(g_2) dg_1 dg_2$$
$$= \int_{R_{\mathbb{A}}^\gamma \setminus (G\times G)_{\mathbb{A}}} (\int_{R_k^\gamma \setminus R_{\mathbb{A}}^\gamma} f(\gamma(r_1, r_2)(g_1', g_2'))\phi_1(r_1 g_1')\phi_2(r_2 g_2') dr_1 dr_2) dg_1' dg_2' \ .$$

If in the inner sum we integrate first over $N_k^\gamma \setminus N_{\mathbb{A}}^\gamma$, the result is a function on $M_k \setminus M_{\mathbb{A}}$ where $M = N^\gamma \setminus R^\gamma$. Hence we may write

$$\int_{R_k^\gamma \setminus R_{\mathbb{A}}^\gamma} f(\gamma(r_1,r_2)(g_1',g_2'))\phi_2(r_1g_1')\phi_2(r_2g_2')dr_1dr_2 =$$

$$= \int_{M_k \setminus M_{\mathbb{A}}} \left(\int_{N_k^\gamma \setminus N_{\mathbb{A}}^\gamma} f(\gamma(n_1,n_2)(m_1,m_2)(g_1',g_2')\right)$$
$$\phi_1(n_1m_1g_1')\phi_2(n_2m_2g_2') \cdot dn_1dn_2)dm_1dm_2 \; .$$

If we now write $N^\gamma = N_1 \times N_2$ with N_i the unipotent radical of some parabolic $P_i \subset G$ (at least one non-trivial), then, since δ is trivial on the unipotent elements of $P_{\mathbb{A}}$, we have the above integral equal to

$$\int_{M_k \setminus M_{\mathbb{A}}} f(\gamma(m_1,m_2)(g_1',g_2')) \left(\int_{N_{1,k} \setminus N_{1,\mathbb{A}}} \phi_1(n_1m_1g_1')dn_1 \right)$$
$$\left(\int_{N_{2,k} \setminus N_{2,\mathbb{A}}} \phi_2(n_2m_2g_2')dn_2 \right) \cdot dm_1dm_2 \; .$$

Since ϕ_1 and ϕ_2 are cusp forms, at least one of the integrals

$$\int_{N_{i,k} \setminus N_{i,\mathbb{A}}} \phi_i(n_im_ig_i')dn_i$$

is identically zero. This implies that for the negligible orbits $I(\gamma) \equiv 0$ and hence they contribute nothing to $L(\omega;\phi_1,\phi_2,f)$ (thus justifying the term negligible). This completes the proof.

Due to the uniqueness of the pairing of π with $\tilde{\pi}$, the global Peterson bilinear form $<,>$ decomposes into a product of local invariant forms $<,>_v$ in the sense that if $\phi_1 = \Pi_v\phi_{1,v} \in \pi$ and $\phi_2 = \Pi_v\phi_{2,v} \in \tilde{\pi}$ are decomposable, then $<\phi_1,\phi_2> = \Pi_v <\phi_{1,v},\phi_{2,v}>_v$. Keeping the basic identity in mind, we now define the local version of our L-functions. For $f \in ind_{P_v}^{H_v}(\omega_v \circ \delta_v), \phi_{1,v} \in \pi_v$, and $\phi_{2,v} \in \tilde{\pi}_v$, define

$$L_v(\omega_v;\phi_{1,v},\phi_{2,v},f_v) = \int_{G_v} f((g,1);\omega_v) < \pi_v(g)\phi_{1,v},\phi_{2,v} >_v dg_v \; .$$

Then as a corollary to the basic identity we have the following.

<u>**Corollary**</u>. The global L-function admits an Euler product given by

$$L(\omega,\phi_1,\phi_2,f) = \Pi_v L_v(\omega_v;\phi_{1,v},\phi_{2,v},f_v)$$

for $\phi_1 \in \pi, \phi_2 \in \tilde{\pi}$, and $f \in ind_{P_{\mathbb{A}}}^{H_{\mathbb{A}}}(\omega \circ \delta)$ all decomposable.

Remark: A priori, one might try to define a global L-function as in (1.1) for $\phi_i \in \pi_i$ with π_i arbitrary irreducible cuspidal automorphic representations. But, by the Basic Identity, these would all be identically zero unless $\pi_2 \simeq \tilde{\pi}_1$ since $< \pi(g)\phi_1, \phi_2 > \equiv 0$ unless $\phi_2 \in \tilde{\pi}_1$.

§2. Explicit constructions for the symplectic, orthogonal, and unitary groups.

In this section we will restrict our attention to the classical groups G of symplectic, orthogonal, or unitary type. In these cases we will explicitly construct a group H, an embedding $i : G \times G \hookrightarrow H$, and a parabolic P of H satisfying the conditions (1) and (2) on the orbits of $G \times G$ in $X = P \setminus H$.

We begin by setting up some common notation. Let k be a global field.

(i) Symplectic groups: Let V be a vector space of even dimension $n = 2m$ over k and let $(\, , \,)$ be a non-degenerate skew-symmetric form on V. Let $G \subset GL(V)$ be the isometries of this form. Then $G = Sp(n)$.

(ii) Orthogonal groups: Let V be a vector space of (arbitrary) dimension n over k and let $(\, , \,)$ denote a non-degenerate symmetric bilinear form on V. Let $G \subset GL(V)$ be the group of isometries of $(\, , \,)$, so $G = 0(n)$.

(iii) Unitary groups: Let K' be a quadratic extension of k. Let V be a vector space over K' of dimension n and let $(\, , \,)$ be a non-degenerate Hermitian form on V with respect to the non-trivial automorphism of K' over k. Then let $G \subset GL(n, K')$ be the group of transformations preserving $(\, , \,)$, so that $G = U(n)$.

Now take G, V and $(\, , \,)$ to be as in any of the cases (i), (ii), or (iii). In cases (i) and (ii) set $K = k$ and in case (iii) set $K = K'$. We will construct H by "doubling the variables" . Let $W = V \oplus V$ and define a form $<,>$ on W by

$$< (v_1, v_2), (v_1', v_2') >= (v_1, v_1') - (v_2, v_2').$$

Then $<,>$ is non-degenerate and of the same type as $(\, , \,)$. The form $<,>$ admits isotropic subspaces of maximal dimension. In fact, if we let $V^d = \{(v, v) \in W\}$ be the image of the diagonal embedding of V in W then $dim_K(V^d) = n = \frac{1}{2} dim_K(W)$ and V^d is isotropic for $<,>$.

Now let $H \subset GL(2n, K)$ be the group of isometries of $<,>$, and let P be the parabolic subgroup of H preserving V^d . There is an embedding $i : G \times G \hookrightarrow H$ by letting

$(v_1, v_2) \cdot i(g_1, g_2) = (v_1 g_1, v_2 g_2)$ for $v_1, v_2 \in V$ and $g_1, g_2 \in G$. Identify $G \times G$ with its image under i .

To show that H, P , and the embedding $i : G \times G \hookrightarrow H$ satisfy conditions (1) and (2) of Section 1 we must investigate the orbit structure of $X = P \setminus H$ under $G \times G$. Let $i_+ : V \hookrightarrow W$ be given by $i_+(v) = (v, o)$ and $i_- : V \hookrightarrow W$ be given by $i_-(v) = (o, v)$. Let $V^{\pm}$ be the image of $i_{\pm}$. Let L be any maximal isotropic subspace of W. Then let $L^+ = L \cap V^+$, $L^- = L \cap V^-, \kappa^+(L) = dim_K(L^+)$, and $\kappa^-(L) = dim_K(L^-)$. Since H acts transitively on the space of maximal isotropic subspaces of W and P stabilizes V^d, then we may view $X = P \setminus H$ as the variety of maximal isotropic subspaces of W .

Lemma 2.1.: Let L be a maximal isotropic subspace of W . Then $\kappa^+(L) = \kappa^-(L) = \kappa(L)$ and $\kappa(L)$ is the only invariant of the $G \times G$ orbit of L in X ; in other words, if $\kappa(L) = \kappa(M)$ for $L, M \in X$, then there exists $g \in G \times G$ such that $Lg = M$.

Proof: Let $\pi^{\pm}$ be the orthogonal projection of W onto $V^{\pm}$. Let $L' = \pi^+(L)$ and $L'' = \pi^-(L)$. Since $L^{\mp}$is the kernel of $\pi^{\pm}$ restricted to L, $dim_K L = dim_K L' + dim_K L^- = dim_K L'' + dim_K L^+$. On the other hand, $L^+ \subset L'$ and L^+ is in the kernel of the form (,) restricted to L' . Since the form is non-degenerate on V^+ there must be a subspace $L_+ \subset V^+$, of the same dimensions as L^+ , which pairs non-degenerately with L^+ . Therefore $L' \oplus L_+ \subset V^+$, so that $dim_K L' + dim_K L^+ \leq dim_K V = dim_K L' + dim_K L^-$, and hence $dim_K L^+ \leq dim_K L^-$. Similarly $dim_K L^- \leq dim_K L^+$, so that $\kappa^+(L) = dim_K L^+ = dim_K L^- = \kappa^-(L)$. Note that this implies that L' is the full orthogonal subspace to L^+ in V^+ and similarly for L'' and L^- in V^- .

Since L^+ is the kernel of the form (,) restricted to L' , (,) induces a non-degenerate form on $L^+ \setminus L'$. Similarly for $L^- \subset L''$. Let $\pi_1 : L' \to L^+ \setminus L'$ and $\pi_2 : L'' \to L^- \setminus L''$ be the projections. Then the isotropic subspace L defines a isometry $g_L : L^+ \setminus L' \to L^- \setminus L''$ by $(\pi_1 v_1) g_L = (\pi_2 v_2)$ iff $(v_1, v_2) \in L$. This is seen to be well-defined and is an isometry since L is totally isotropic in W. Furthermore, the spaces L', L^+, L'', L^- and the isometry g_L completely determine L, for $L = \{(v_1, v_2) : v_1 \in L', v_2 \in L''$ and $(\pi_1 v_1) g_L = \pi_2 v_2\}$.

That $\kappa(L)$ is an invariant of the $G \times G$ orbit of L in X is evident, since $(L(g_1, g_2))^+ = L^+ g_1$. Now we will show that if L and M are totally isotropic subspaces of W with $\kappa(L) = \kappa(M)$ then there exists $g = (g_1, g_2) \in G \times G$ such that $Lg = M$. Since L^+ and M^+ are isotropic in V^+ of the same dimension, there is $L^+ g_1 \doteq M^+$. Similarly there is

$g_2 \in G$ such that $L^- g_2 = M^-$. So replacing L by $L(g_1, g_2)$ we may assume $L^+ = M^+$ and $L^- = M^-$. Then since L' is the orthocomplement of L^+ , and the same is true for M' , we have $L' = M'$. Similarly $L'' = M''$. Then L and M both define isometries $g_L, g_M : L^+ \setminus L' \to L^- \setminus L''$. These will differ by an isometry γ of $L^- \setminus L''$, i.e., $g_L = g_M \gamma$ with $\gamma : L^- \setminus L'' \to L^- \setminus L''$. γ may be lifted to an isometry γ'' of L'' satisfying $\pi_2(v_2\gamma'') = (\pi_2 v_2)\gamma$ and this may be extended, via Witt's theorem, to an isometry γ'' of V^-. We claim that $L(1,\gamma'')^{-1} = M$, with $(1,\gamma'') \in G \times G$. We have $(v_1, v_2) \in L$ iff $(\pi_1 v_1) g_L = (\pi_2 v_2)$. Therefore $L(1,\gamma'')^{-1} = \{(v_1, v_2) \in L' \times L'' : (\pi_1 v_1) g_L = \pi_2(v_2 \gamma'')\}$. But $g_L \gamma^{-1} = g_M$. Therefore $(v_1, v_2) \in L(1,\gamma'')^{-1}$ iff $(\pi_1 v_1) g_M = (\pi_2 v_2)$ iff $(v_1, v_2) \in M$. This completes the proof.

Now for $0 \leq d \leq n$, let X_d be the $G \times G$ orbit in X of maximal isotropic subspaces L with $\kappa(L) = d$. Then since $\kappa(V^d) = 0$ we have V^d in the orbit X_0 . The stabilizer of V^d in $G \times G$ is $(G \times G) \cap P$. On the other hand, an element $(g_1, g_2) \in G \times G$ stabilizes V^d iff $vg_1 = vg_2$ for all $v \in V$, i.e., iff $g_1 = g_2$. So indeed $G^d = (G \times G) \cap P$ and condition (1) is satisfied.

To show that condition (2) is satisfied, we must show that if $d > 0$ then the orbit X_d is negligible. So fix $d > 0$ and let $L \in X_d$. Let P^+ be the parabolic subgroup of G preserving the flag $V \supset L' \supset L^+$ and P^- the parabolic subgroup of G preserving the flag $V \supset L'' \supset L^-$. Since $d > 0$ these are proper parabolics. Let $N^\pm$ be the unipotent radical of $P^\pm$ so that $N = N^+ \times N^-$ is then the unipotent radical of the proper parabolic $P^+ \times P^-$ of $G \times G$. Now let R be the stabilizer of L in $G \times G$. R can be characterized as the pairs $(g_1, g_2) \in P^+ \times P^-$ such that $(\pi_1(v_1 g_1)) g_L = \pi_2(v_2 g_2)$. (Recall that $\pi_1 : L' \to L^+ \setminus L'$ and $\pi_2 : L'' \to L^- \setminus L''$ are the canonical projections.) Since N is normal in $P^+ \times P^-$ we need only show that $N \subset R$. But by definition, N^+ induces the identity on $L^+ \setminus L'$ and N^- induces the identity on $L^- \setminus L''$. So for $(v_1, v_2) \in L' \times L''$ and $(n_1, n_2) \in N$ we have $\pi_1(v_1 n_1) = \pi_1(v_1)$ and $\pi_2(v_2 n_2) = \pi_2(v_2)$. So $(\pi_1(v_1 n_1)) g_L = (\pi_1(v_1)) g_L = \pi_2(v_2) = \pi_2(v_2 n_2)$. Then $N \subset R$. This shows that for $d > 0$ the orbit X_d is negligible. Therefore we have proved the following proposition.

Proposition 2.1: The choices of group H, parabolic P and embedding $i : G \times G \to H$ above satisfy conditions (1) and (2) of Section 1.

§3. Explicit constructions for $PGL(n)$. Connections with the work of Godement and Jacquet.

Let D be a central simple division algebra of degree m over k and let $G = PGL(n, D)$. In this section we will construct a group H, parabolic subgroup $P \subset H$ and an embedding $i : G \times G \hookrightarrow H$ satisfying conditions (1) and (2) of Sect. 1. In this situation, the L-functions of Section 1 will be the same as the zeta functions considered by Godement and Jacquet in [G-J].

3.1. Let $V = M(n, D)$. As a vector space over k, V has dimension $N = n^2m^2$. There is a natural action of $GL(n, D) \times GL(n, D)$ on V by $x \cdot (g_1, g_2) = g_2^{-1}xg_1$ for $g_i \in GL(n, D)$ and $x \in M(n, D)$. This gives a homomorphism $GL(n, D) \times GL(n, D) \to GL(N, k)$ which will induce an embedding $i : G \times G \hookrightarrow PGL(N, k)$. We will take $H = PGL(N, k)$ and $i : G \times G \hookrightarrow H$ this embedding. Identify $G \times G$ with its image. Let $e_0 \in V$ correspond to the identity $1_n \in M(n, D)$. Take P to be the parabolic subgroup of H stabilizing the k-line through e_0 .

To show that conditions (1) and (2) of Sect. 1 are satisfied, we must determine the orbit structure of $X = P \setminus H$ under the action of $G \times G$. Since P stabilizes a k-line in V, X is the variety of k-lines in V . For $x \in V$ we will let $< x >$ denote the k-line spanned by x. Let $W = D^n$ be a vector space over D considered as a space of row vectors. W is a D-module under left multiplication and a $M(n, D)$ module under right matrix multiplication. As a vector space over k, $dim_k(W) = nm^2$. For each $x \in V$ we may define an invariant of the $G \times G$ orbit of $< x >$ in X as follows. Viewing x as an element of $M(n, D)$, Wx will be a subspace of W. If $y = \lambda x$ with $\lambda \in k^v$ then $Wy = Wx$ and hence Wx depends only on the span $< x >$ of x. Then define $\kappa(x) = dim_k(Wx)$.

<u>**Lemma 3.1**</u>: For $x \in V$, $\kappa(x)$ is the only invariant of the $G \times G$ orbit of $< x >$ in X, in other words if $\kappa(x_1) = \kappa(x_2)$, $x_1, x_2 \in V$ then there exists a $g \in G \times G$, such that $< x_1 > g =< x_2 >$.

<u>**Proof:**</u> We have already seen that $\kappa(x)$ depends only on the span $< x >$ of x. To show that $\kappa(x)$ is an invariant of the $G \times G$ orbit it will suffice to show that for $g_1, g_2 \in GL(n, D)$, $\kappa(g_2^{-1}xg_1) = \kappa(x)$. But this is clear since $\kappa(g_2^{-1}xg_1) = dim_k(Wg_2^{-1}xg_1) =$