AMERICAN MATHEMATICAL SOCIETY COLLOQUIUM PUBLICATIONS

Volume XVII

AMERICAN MATHEMATICAL SOCIETY
COLLOQUIUM PUBLICATIONS
VOLUME XVII

LECTURES ON MATRICES

BY

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PUBLISHED BY THE
AMERICAN MATHEMATICAL SOCIETY
501 West 116th Street, New York
1934

PREFACE

This book contains lectures on matrices given at Princeton University at various times since 1920. It was my intention to include full notes on the history of the subject, but this has proved impossible owing to circumstances beyond my control, and I have had to content myself with very brief notes (see Appendix I). A bibliography is given in Appendix II. In compiling it, especially for the period of the last twenty-five years, there was considerable difficulty in deciding whether to include certain papers which, if they had occurred earlier, would probably have found a place there. In the main, I have not included articles which do not use matrices as an algebraic calculus, or whose interest lies in some other part of mathematics rather than in the theory of matrices; but consistency in this has probably not been attained.

Since these lectures have been prepared over a somewhat lengthy period of time, they owe much to the criticism of many friends. In particular, Professor A. A. Albert and Dr. J. L. Dorroh read most of the MS making many suggestions, and the former gave material help in the preparation of the later sections of Chapter X.

J. H. M. Wedderburn.

Princeton, N. J.,
July 20, 1934.

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17. *J. H. M. Wedderburn*, Lectures on Matrices. 1934. 200 pp. \$3.00.

AMERICAN MATHEMATICAL SOCIETY

New York City: 501 West 116th Street

Menasha, Wis., 450 Ahnaip Street

Cambridge, England, 1 Trinity Street, Bowes and Bowes

Berlin, Germany, Unter den Linden 68, Hirschwaldsche Buchhandlung

Bologna, Italy, Nicola Zanichelli

Photo-Lithoprint Reproduction
EDWARDS BROTHERS, INC.
Lithoprinters
ANN ARBOR, MICHIGAN, U.S.A.
1949

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APPENDIX II

CORRIGENDA

page 4, line 9 from top: on second Σ read j for p

page 6, Theorem 1, add: *and conversely, if a matrix is commutative with every other matrix, it is a scalar matrix.*

page 7, line 12 from foot; for first and third A read $|A|$

page 11, lines 10, 11 from foot: for $(Q'Q)$ read $(Q'Q)^{-1}$

page 13, line 9 from foot: for Sg_ix read $S\bar{g}_ix$

page 13, line 6 from foot: for $\bar{g}_{s+1}$ read $\bar{g}'_{s+1}$

page 14, line 3 from top: before $a_{ji}e_j$ read Σ.

page 18, line 8 from foot: for j read γ_j

page 20, line 8 from foot: for $r + 1$ read $r - 1$

page 30, line 13 from top: for $=$ read $-$

page 31, line 4 from foot: for second x_i read xe_i; add $e_1 + e_2 = 1$

page 42, equation (16): for 1 read -1

page 54, line 14 from foot; for (12) read (13)

page 54, line 6 from foot: for (14) read (15)

page 54, line 3 from foot: for (15) read (16)

page 54, line 2 from foot: for (13) read (14)

page 56, line 12 from top: correct term after Σ to read $k_{i,\beta+j}z_i$

page 67, lines 4, 5, 6, 7: the exponent n on the second last C should read $\binom{n}{r}$.

page 68, line 11 from foot: before Σ read $(-1)^r$

page 68, line 8 from foot: before $|\mathfrak{A}|$ read $(-1)^r$

page 74, line 11 from foot: for $r = 1$ read $r = 3$

page 81, line 4 from foot: for $1/\beta_1!$ read $\beta_1!$ with similar change in last line

page 84, line 13 from foot: interchange i and j.

page 85, line 8 from foot: for $\mathfrak{F}_1$ read $\bar{\mathfrak{F}}_1$

page 86, line 7 from top: for first e_1 read e_i

page 92, line 11 from foot: delete from "and if" to end of paragraph

page 101, line 6 from foot: after hermitian insert $A = A'$

page 103, line 4 from foot: delete first 0; for $q = t + 1$ read $q = s + 1$

page 112, equation (23): for { } read [ ]

page 116, line 7 from top: add Every power series converges when x is nilpotent.

page 119, line 9 from top: for "at least . . . first" read "the H.C.F. of the t's is relatively prime to m"

page 122, line 4 from foot: multiply bracket by $\epsilon^{\mu i}$ and delete same inside

page 122, equation (30): for g_{ij} read $p_{ij} = \epsilon^{-\mu i} g_{ij}$

page 123, lines 2 and 3 from top: for g_{ij} read p_{ij}

page 123, equations (32) and (33): for π read 2π

page 125, line 4 from top: read $\alpha_1^{r_1}(\lambda), \alpha_2^{r_2}(\lambda), \cdots, \alpha_k^{r_k}(\lambda)$

page 126, line 13 from top: for $|\ |$ read **| |**

page 126, equation (45): for first α read a

page 129, equation (63): in first term the bars should be heavy

page 129, line 5 from foot: for $|x|$ read **|**x**|**

page 134, line 6 from top: multiply right side of equation by 2

page 136, line 10 from top: for ξ_r read ξ_{r-1}

page 137, equation (103): read $\dot{p} = -\partial_q \mathfrak{H}$

page 144, equation (24): read $x'axa^{-1}$

page 156, line 6 from top: for second = read $\leq$ and add "$\leq A$, whence $A = \Sigma A_{ij}$"

page 164, line 8 from top: for primitive read minimal

page 164, line 7 from foot: for invariant read semi-invariant

page 164, last line: before "complete" insert "suitably chosen"

page 166, line 10 from foot: for equivalent read invariant

page 166, line 5 from foot: for first B_2 read B_1

page 167, Theorem 9: for $j \neq k$ read $i \neq t$

page 171, line 5 from top: delete 80

CHAPTER I

MATRICES AND VECTORS

1.01 **Linear transformations and vectors.** In a set of linear equations

$$\begin{aligned}
\eta'_1 &= a_{11}\eta_1 + a_{12}\eta_2 + \cdots + a_{1n}\eta_n \\
\eta'_2 &= a_{21}\eta_1 + a_{22}\eta_2 + \cdots + a_{2n}\eta_n \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
\eta'_n &= a_{n1}\eta_1 + a_{n2}\eta_2 + \cdots + a_{nn}\eta_n
\end{aligned}$$

or

(1) $$\eta'_i = \sum_{j=1}^{n} a_{ij}\eta_j \qquad (i = 1, 2, \cdots, n)$$

the quantities $\eta_1, \eta_2, \cdots, \eta_n$ may be regarded as the coordinates of a point P in n-space and the point $P'(\eta'_1, \eta'_2, \cdots, \eta'_n)$ is then said to be derived from P by the *linear homogeneous transformation* (1). Or, in place of regarding the η's as the coordinates of a point we may look on them as the components of a vector y and consider (1) as defining an operation which transforms y into a new vector y'. We shall be concerned here with the properties of such transformations, sometimes considered abstractly as entities in themselves, and sometimes in conjunction with vectors.

To prevent misconceptions as to their meaning we shall now define a few terms which are probably already familiar to the reader. By a *scalar* or number we mean an element of the field in which all coefficients of transformations and vectors are supposed to lie; unless otherwise stated the reader may assume that a scalar is an ordinary number real or complex.

A *vector*[1] of order n is defined as a set of n scalars $(\xi_1, \xi_2, \cdots, \xi_n)$ given in a definite order. This set, regarded as a single entity, is denoted by a single symbol, say x, and we write

$$x = (\xi_1, \xi_2, \cdots, \xi_n).$$

The scalars $\xi_1, \xi_2, \cdots, \xi_n$ are called the *coordinates* or *components* of the vector.

If $y = (\eta_1, \eta_2, \cdots, \eta_n)$ is also a vector, we say that $x = y$ if, and only if, corresponding coordinates are equal, that is, $\xi_i = \eta_i$ $(i = 1, 2, \cdots, n)$. The vector

$$z = (\zeta_1, \zeta_2, \cdots, \zeta_n) = (\xi_1 + \eta_1, \xi_2 + \eta_2, \cdots, \xi_n + \eta_n)$$

is called the *sum* of x and y and is written $x + y$; it is easily seen that the operation of addition so defined is commutative and associative, and it has a unique inverse if we agree to write 0 for the vector $(0, 0, \cdots, 0)$.

[1] In chapter 5 we shall find it convenient to use the name *hypernumber* for the term vector which is then used in a more restricted sense, which, however, does not conflict with the use made of it here.

If ρ is a scalar, we shall write

$$\rho x = x\rho = (\rho\xi_1, \rho\xi_2, \cdots, \rho\xi_n).$$

This is the only kind of multiplication we shall use regularly in connection with vectors.

1.02 Linear dependence. In this section we shall express in terms of vectors the familiar notions of linear dependence.[2] If $x_1, x_2, \cdots, x_r$ are vectors and $\omega_1, \omega_2, \cdots, \omega_r$ scalars, any vector of the form

$$x = \omega_1 x_1 + \omega_2 x_2 + \cdots + \omega_r x_r \tag{2}$$

is said to be *linearly dependent* on $x_1, x_2, \cdots, x_r$; and these vectors are called linearly independent if an equation which is reducible to the form

$$0 = \omega_1 x_1 + \omega_2 x_2 + \cdots + \omega_r x_r$$

can only be true when each $\omega_i = 0$. Geometrically the r vectors determine an r-dimensional subspace of the original n-space and, if $x_1, x_2, \cdots, x_r$ are taken as the coordinate axes, $\omega_1, \omega_2, \cdots, \omega_r$ in (2) are the coordinates of x.

We shall call the totality of vectors x of the form (2) the *linear set* or *subspace* $(x_1, x_2, \cdots, x_r)$ and, when $x_1, x_2, \cdots, x_r$ are linearly independent, they are said to form a *basis* of the set. The number of elements in a basis of a set is called the *order* of the set.

Suppose now that $(x_1, x_2, \cdots, x_r)$, $(y_1, y_2, \cdots, y_s)$ are bases of the same linear set and assume $s \geq r$. Since the x's form a basis, each y can be expressed in the form

$$y_i = a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{ir}x_r \qquad (i = 1, 2, \cdots, s) \tag{3}$$

and, since the y's form a basis, we may set

$$x_i = b_{i1}y_1 + b_{i2}y_2 + \cdots + b_{is}y_s \qquad (i = 1, 2, \cdots, r)$$

and therefore from (3)

$$y_i = \sum_{j=1}^{r} a_{ij}x_j = \sum_{j=1}^{r} a_{ij}\sum_{k=1}^{s} b_{jk}y_k = \sum_{k=1}^{s} c_{ik}y_k, \tag{4}$$

where $c_{ik} = \sum_{j=1}^{r} a_{ij}b_{jk}$, which may also be written

$$c_{ik} = \sum_{j=1}^{s} a_{ij}b_{jk} \qquad (i = 1, 2, \cdots, s) \tag{5}$$

if we agree to set $a_{ij} = 0$ when $j > r$. Since the y's are linearly independent, (4) can only hold true if $c_{ii} = 1$, $c_{ik} = 0$ $(i \neq k)$ so that the determinant

[2] See for instance Bôcher, *Introduction to Higher Algebra*, p. 34.

$|c_{ik}| = 1$. But from the rule for forming the product of two determinants it follows from (5) that $|c_{ik}| = |a_{ik}||b_{ik}|$ which implies (i) that $|a_{ik}| \neq 0$ and (ii) that $r = s$, since otherwise $|a_{ik}|$ contains the column $a_{i,\, r+1}$ each element of which is 0. The order of a set is therefore independent of the basis chosen to represent it.

It follows readily from the theory of linear equations (or from §1.11 below) that, if $|a_{ij}| \neq 0$ in (3), then these equations can be solved for the x's in terms of the y's, so that the conditions established above are sufficient as well as necessary in order that the y's shall form a basis.

If e_i denotes the vector whose ith coordinate is 1 and whose other coordinates are 0, we see immediately that we may write

$$x = \xi_1 e_1 + \xi_2 e_2 + \cdots + \xi_n e_n$$

in place of $x = (\xi_1, \xi_2, \cdots, \xi_n)$. Hence $e_1, e_2, \cdots, e_n$ form a basis of our n-space. We shall call this the *fundamental basis* and the individual vectors e_i the fundamental *unit vectors.*

If $x_1, x_2, \cdots, x_r (r < n)$ is a basis of a subspace of order r, we can always find $n-r$ vectors $x_{r+1}, \cdots, x_n$ such that $x_1, x_2, \cdots, x_n$ is a basis of the fundamental space. For, if x_{r+1} is any vector not lying in $(x_1, x_2, \cdots, x_r)$, there cannot be any relation

$$\omega_1 x_1 + \omega_2 x_2 + \cdots + \omega_r x_r + \omega_{r+1} x_{r+1} = 0$$

in which $\omega_{r+1} \neq 0$ (in fact every ω must be 0) and hence the order of $(x_1, x_2, \cdots, x_r, x_{r+1})$ is $r + 1$. Since the order of $(e_1, e_2, \cdots, e_n)$ is n, a repetition of this process leads to a basis $x_1, x_2, \cdots, x_r, \cdots, x_n$ of order n after a finite number of steps; a suitably chosen e_i may be taken for x_{r+1}. The $(n-r)$-space $(x_{r+1}, \cdots, x_n)$ is said to be *complementary* to $(x_1, x_2, \cdots, x_r)$; it is of course not unique.

1.03 Linear vector functions and matrices. The set of linear equations given in §1.01, namely,

$$\eta'_i = \sum_{j=1}^{n} a_{ij}\eta_j \qquad (i = 1, 2, \cdots, n) \tag{6}$$

define the vector $y' = (\eta'_1, \eta'_2, \cdots, \eta'_n)$ as a linear homogeneous function of the coordinates of $y = (\eta_1, \eta_2, \cdots, \eta_n)$ and in accordance with the usual functional notation it is natural to write $y' = A(y)$; it is usual to omit the brackets and we therefore set in place of (6)

$$y' = Ay.$$

The function or operator A when regarded as a single entity is called a matrix; it is completely determined, relatively to the fundamental basis, when the n^2 numbers a_{ij} are known, in much the same way as the vector y is determined by its coordinates. We call the a_{ij} the *coordinates* of A and write

$$A = \begin{Vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{Vmatrix} \tag{7}$$

or, when convenient, $A = \| a_{ij} \|$. It should be noted that in a_{ij} the first suffix denotes the row in which the coordinate occurs while the second gives the column.

If $B = \| b_{ij} \|$ is a second matrix, $y'' = A(By)$ is a vector which is a linear vector homogeneous function of y, and from (6) we have

$$\eta_i'' = \sum_{p=1}^{n} a_{ip} \sum_{p=1}^{n} b_{pj}\eta_j = \sum_{j=1}^{n} d_{ij}\eta_j,$$

where

$$d_{ij} = \sum_{p=1}^{n} a_{ip}b_{pj}. \tag{8}$$

The matrix $D = \| d_{ij} \|$ is called the *product* of A into B and is written AB. The form of (8) should be carefully noted; in it each element of the ith row of A is multiplied into the corresponding element of the jth column of B and the terms so formed are added. Since the rows and columns are not interchangeable, AB is in general different from BA; for instance

$$\begin{Vmatrix} 1 & 0 \\ 2 & 1 \end{Vmatrix} \begin{Vmatrix} a & b \\ c & d \end{Vmatrix} = \begin{Vmatrix} a & b \\ 2a + c & 2b + d \end{Vmatrix}$$

$$\begin{Vmatrix} a & b \\ c & d \end{Vmatrix} \begin{Vmatrix} 1 & 0 \\ 2 & 1 \end{Vmatrix} = \begin{Vmatrix} a + 2b & b \\ c + 2d & d \end{Vmatrix}.$$

The product defined by (8) is associative; for if $C = \| c_{ij} \|$, the element in the ith row and jth column of $(AB)C$ is

$$\sum_{q=1}^{n} \left(\sum_{p=1}^{n} a_{ip}b_{pq} \right) c_{qj} = \sum_{p=1}^{n} a_{ip} \left(\sum_{q=1}^{n} b_{pq}c_{qj} \right)$$

and the term on the right is the (i, j) coordinate of $A(BC)$.

If we add the vectors Ay and By, we get a vector whose ith coordinate is (cf. (6))

$$\eta_i' = \sum_{j=1}^{n} a_{ij}\eta_j + \sum_{j=1}^{n} b_{ij}\eta_j = \sum_{j=1}^{n} c_{ij}\eta_j$$

where $c_{ij} = a_{ij} + b_{ij}$. Hence $Ay + By$ may be written Cy where $C = \| c_{ij} \|$. We define C to be the *sum* of A and B and write $C = A + B$; two matrices are then added by adding corresponding coordinates just as in the case of vectors. It follows immediately from the definition of sum and product that

$$A + B = B + A, \qquad (A + B) + C = A + (B + C),$$

$$A(B + C) = AB + AC, \qquad (B + C)A = BA + CA,$$

$$A(x + y) = Ax + Ay,$$

A, B, C being any matrices and x, y vectors. Also, if k is a scalar and we set $y' = Ay$, $y'' = ky'$, then

$$y'' = ky' = kA(y) = A(ky)$$

or in terms of the coordinates

$$\eta_i'' = \sum_j k a_{ij} \eta_j.$$

Hence kA may be interpreted as the matrix derived from A by multiplying each coordinate of A by k.

On the analogy of the unit vectors e_i we now define the *fundamental unit matrices* e_{ij} $(i, j = 1, 2, \cdots, n)$. Here e_{ij} is the matrix whose coordinates are all 0 except the one in the ith row and jth column whose value is 1. Corresponding to the form $\Sigma \xi_i e_i$ for a vector we then have

$$A = \sum_{i,j=1}^{n} a_{ij} e_{ij}. \tag{9}$$

Also from the definition of multiplication in (8)

$$e_{ij} e_{jk} = e_{ik}, \qquad e_{ij} e_{pq} = 0, \qquad (j \neq p) \tag{10}$$

a set of relations which might have been made the basis of the definition of the product of two matrices. It should be noted that it follows from the definition of e_{ij} that

$$e_{ij} e_j = e_i, \qquad e_{ij} e_k = 0 \qquad (j \neq k), \tag{11}$$

$$Ae_k = \sum_{i,j} a_{ij} e_{ij} e_k = \sum_i a_{ik} e_i. \tag{12}$$

Hence the coordinates of Ae_k are the coordinates of A that lie in the kth column.

1.04 **Scalar matrices.** If k is a scalar, the matrix K defined by $Ky = ky$ is called a *scalar matrix*; from (1) it follows that, if $K = \| k_{ij} \|$, then $k_{ii} = k$ $(i = 1, 2, \cdots, n)$, $k_{ij} = 0$ $(i \neq j)$. The scalar matrix for which $k = 1$ is called the identity matrix of order n; it is commonly denoted by I but, for reasons explained below, we shall here usually denote it by 1, or by 1_n if it is desired to indicate the order. When written at length we have

$$1_n = \left\| \begin{array}{cccc} 1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{array} \right\|, \quad K = \left\| \begin{array}{cccc} k & & & \\ & k & & \\ & & \ddots & \\ & & & k \end{array} \right\|$$

A convenient notation for the coordinates of the identity matrix was introduced by Kronecker: if δ_{ij} is the numerical function of the integers i, j defined by

$$(13) \qquad \delta_{ii} = 1, \qquad \delta_{ij} = 0 \qquad (i \neq j),$$

then $1_n = \| \delta_{ij} \|$. We shall use this Kronecker delta function in future without further comment.

THEOREM 1. *Every matrix is commutative with a scalar matrix.*

Let k be the scalar and $K = \| k_{ij} \| = \| k\delta_{ij} \|$ the corresponding matrix. If $A = \| a_{ij} \|$ is any matrix, then from the definition of multiplication

$$KA = \left\| \sum_p k_{ip}a_{pj} \right\| = \left\| \sum_p k\delta_{ip}a_{pj} \right\| = \| ka_{ij} \|$$

$$AK = \left\| \sum_p a_{ip}k_{pj} \right\| = \left\| \sum_p ka_{ip}\delta_{pj} \right\| = \| ka_{ij} \|$$

so that $AK = KA$.

If k and h are two scalars and K, H the corresponding scalar matrices, then $K + H$ and KH are the scalar matrices corresponding to $k + h$ and kh. Hence the *one-to-one* correspondence between scalars and scalar matrices is maintained under the operations of addition and multiplication, that is, the two sets are simply isomorphic with respect to these operations. So long therefore as we are concerned only with matrices of given order, there is no confusion introduced if we replace each scalar by its corresponding scalar matrix, just as in the theory of ordinary complex numbers, $(a, b) = a + bi$, the set of numbers of the form $(a, 0)$ is identified with the real continuum. We shall therefore as a rule denote $\| \delta_{ij} \|$ by 1 and $\| k\delta_{ij} \|$ by k.

1.05 Powers of a matrix; adjoint matrices. Positive integral powers of $A = \| a_{ij} \|$ are readily defined by induction; thus

$$A^2 = A \cdot A, \qquad A^3 = A \cdot A^2, \cdots, \qquad A^m = A \cdot A^{m-1}.$$

With this definition it is clear that $A^rA^s = A^{r+s}$ for any positive integers r, s. Negative powers, however, require more careful consideration.

Let the determinant formed from the array of coefficients of a matrix be denoted by

$$| A | = \text{det. } A$$

and let α_{qp} be the cofactor of a_{pq} in A, so that from the properties of determinants

$$\sum_p a_{ip}\alpha_{pj} = | A | \, \delta_{ij} = \sum_p \alpha_{ip}a_{pj} \qquad (i, j = 1, 2, \cdots, n). \tag{14}$$

The matrix $\| \alpha_{ij} \|$ is called the *adjoint* of A and is denoted by adj A. In this notation (14) may be written

$$A(\text{adj } A) = | A | = (\text{adj } A)A, \tag{15}$$

so that a matrix and its adjoint are commutative.

If $| A | \neq 0$, we define A^{-1} by

$$A^{-1} = | A |^{-1} \text{ adj } A. \tag{16}$$

Negative integral powers are then defined by $A^{-r} = (A^{-1})^r$; evidently $A^{-r} = (A^r)^{-1}$. We also set $A^0 = 1$, but it will appear later that a different interpretation must be given when $| A | = 0$. Since $AB \cdot B^{-1}A^{-1} = A \cdot BB^{-1} \cdot A^{-1} = AA^{-1} = 1$, the reciprocal of the product AB is

$$(AB)^{-1} = B^{-1}A^{-1}.$$

If A and B are matrices, the rule for multiplying determinants, when stated in our notation, becomes

$$| AB | = | A | \, | B |.$$

In particular, if $AB = 1$, then $| A | \, | B | = 1$; hence, if $| A | = 0$, there is no matrix B such that $AB = 1$ or $BA = 1$. The reader should notice that, if k is a scalar matrix of order n, then $| k | = k^n$.

If $A = 0$, A is said to be *singular*; if $A \neq 0$, A is *regular* or non-singular. When A is regular, A^{-1} is the only solution of $AX = 1$ or of $XA = 1$. For, if $AX = 1$, then

$$A^{-1} = A^{-1} \cdot 1 = A^{-1}AX = X.$$

If $AX = 0$, then either $X = 0$ or A is singular; for, if A^{-1} exists,

$$0 = A^{-1}Ax = X.$$

If $A^2 = A \neq 0$, then A is said to be *idempotent*; for example e_{11} and $\begin{Vmatrix} 4 & -2 \\ 6 & -3 \end{Vmatrix}$ are idempotent. A matrix a power of which is 0 is called *nilpotent*. If the lowest power of A which is 0 is A^r, r is called the *index* of A; for example, if $A = e_{12} + e_{23} + e_{34}$, then

$$A^2 = e_{13} + e_{24}, \qquad A^3 = e_{14}, \qquad A^4 = 0,$$

so that the index of A in this case is 4.