Lecture Notes in Mathematics

Edited by A. Dold and B. Eckmann

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J. L. Bueso P. Jara
B. Torrecillas (Eds.)

Ring Theory

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EDITORIAL

These proceedings contain papers presented in the meeting in "Ring Theory" celebrated in Granada (SPAIN). Not all the lectures and communications appear here, some of them were given by invited speakers, who were unable to attend, and others are being published elsewhere.

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STABLE RANGE OF ALEPH-NOUGHT-CONTINUOUS REGULAR RINGS

PERE ARA

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ABSTRACT. *In this paper we show that if* R *is a right* $\aleph_0$*-continuous regular ring, then the set of possible values for the stable range of* R, sr(R), *is* $\{1,2,\infty\}$. *Further,* sr(R) = 1 *if and only if* R *is directly finite, and* sr(R) $\leq$ 2 *if and only if* R *is an Hermite ring.*

All rings considered in this paper are associative with 1, and all modules are unital.

A ring R is said to be <u>regular</u> if for every $a \in R$ there exists an element $b \in R$ such that $a = aba$.

Let R be any ring. A n-row $x=(x_1,\ldots,x_n) \in R^n$ is <u>unimodular</u> if $x_1R + \ldots + x_nR = R$, and x is <u>reducible</u> if there exist $y_1,\ldots,y_{n-1} \in R$ such that the (n - 1)-row $(x_1 + x_ny_1,\ldots,x_{n-1} + x_ny_{n-1})$ is unimodular. R is said to have <u>stable range</u> n, sr(R) = n, if n is the least positive integer such that every unimodular (n+1)-row is reducible; the stable range of R is ∞ if there does not exist any positive integer with this property.

Recall [4, p. 465] that a ring R is <u>right</u> (<u>left</u>) <u>Hermite</u> if every 1x2 (2x1) matrix admits diagonal reduction. In other words,

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R is right Hermite if given a 2-row $x \in R^2$ there exists an invertible 2x2 matrix Q such that $xQ = (*,0)$. A right and left Hermite ring is called an Hermite ring

A ring R is said to be unit-regular if for every $x \in R$ there exists a unit $u \in R$ such that $x = xux$. The unit-regular rings are precisely those regular rings which have stable range 1 [2, Prop.4.12] .

We shall use the following results [5] .

THEOREM 1. (Menal-Moncasi; [5, Thm. 3(b)]). *Let M be a right R-module such that* $S = \mathrm{End}_R(M)$ *is a regular ring. Then the following conditions are equivalent:*

(i) S *has stable range* $\leqslant n$.

(ii) $M^n \oplus B \cong M \oplus C$ *implies that* B *is isomorphic to a direct summand of* C , *for all right* R-*modules* B,C .

(iii) For every $x \in S$, $M^n \oplus x(M) \cong M \oplus C$ *implies that* x(M) *is isomorphic to a direct summand of* C , *for all right module* C. □

THEOREM 2. (Menal-Moncasi; [5, Thm. 9]). *Let M be a right R-module such that* $S = \mathrm{End}_R(M)$ *is a regular ring. Then the following statements are equivalent:*

(i) S *is left Hermite.*

(ii) S *is right Hermite.*

(iii) $M^2 \oplus B \cong M \oplus C$ *implies* $M \oplus B \cong C$, *for all right* R-*modules* B,C .

(iv) For every $x \in S$, $M^2 \oplus x(M) \cong M \oplus C$ *implies* $M \oplus x(M) \cong C$ *for all right* R-*module* C.

(v) Every matrix over S *admits diagonal reduction.* □

It follows from these results that Hermite regular rings have stable range $\leqslant 2$. It is an open question whether regular rings with stable range $\leqslant 2$ are Hermite rings (see [6, Problema 2]).

Let R be a regular ring. R is said to be right $\aleph_0$-continuous provided the lattice $L(R_R)$ of principal right ideals of R is upper $\aleph_0$-continuous, i.e., every countable subset of $L(R_R)$ has a supremum in $L(R_R)$, and

$$A \wedge \left(\bigvee_{n=1}^{\infty} B_n \right) = \bigvee_{n=1}^{\infty} (A \wedge B_n)$$

for every $A \in L(R_R)$ and every countable ascending chain $B_1 \leqslant B_2 \leqslant \ldots$ in $L(R_R)$. It is shown in [2,Corollary 14.4] that R is right $\aleph_0$-continuous if and only if every countably generated right ideal of R is essential in a principal right ideal of R.

Recall that a ring R is said to be <u>directly finite</u> if, for $x,y \in R$, $xy = 1$ implies $yx = 1$. It is a simple exercise to show that any unit-regular ring is directly finite. On the other hand there are examples of directly finite regular rings that are not unit-regular [2,Example 5.10]. One of the key results on right $\aleph_0$-continuous regular rings is a theorem of Goodearl which asserts that any directly finite right $\aleph_0$-continuous regular ring is unit-regular [3,Theorem 1.4]. So for this class of rings, $sr(R) = 1$ iff R is directly finite. In this note we give a complete characterization of the stable range of right $\aleph_0$-continuous regular rings. In particular we show that a right $\aleph_0$-continuous regular ring can only have stable range $1,2$ or ∞. It is an open question whether the latter is true for arbitrary regular rings (cf.[6,Problema 1]).

First we observe that there exist examples of right $\aleph_0$-continuous regular rings with stable range $1,2$ or ∞. It is a simple matter to give examples with stable range 1 or ∞. For instance regular right self-injective rings are right $\aleph_0$-continuous and there are well-known examples of regular right self-injective rings with stable range 1 or ∞. To construct examples with stable range 2 we proceed as follows. Let F be any field, let V be an F-vector space such that $\dim_F(V) > \aleph_0$, and let τ be a cardinal number such that $\aleph_0 \leqslant \tau < \dim_F(V)$. Set $S = \mathrm{End}_F(V)$ and $M = \{x \in S: \dim_F x(V) \leqslant \tau\}$. Finally set $R = F + M$. Then it is easily seen that R is a right $\aleph_0$-continuous regular ring and, by using the argument in [5,p.39] we have that R is a regular Hermite ring.

We shall use the following results from [1]. For a regular ring R define $I = \{x \in R : R \oplus xR \lesssim R\}$. Then by [1,Prop.2.1], I is a two-sided ideal of R such that R/I is directly finite; moreover if H is any two-sided ideal of R with R/H directly finite, then $I \leqslant H$. If R is right $\aleph_0$-continuous then, by [1,Thm.2.7], R/I is a directly finite right $\aleph_0$-continuous regular ring.

We will need the following lemma. For the reader's convenience we include a proof.

LEMMA 3. (Moncasi [6]). *Let* R *be a regular ring and let* M *be a two-sided ideal of* R *such that* R/M *is unit-regular. If* $R \cong (1 - e)R$ *for every idempotent* $e \in M$, *then* R *is an Hermite ring.*

Proof. By Theorem 2 it suffices to prove that $R^2 \oplus fR \cong R \oplus C$ implies $R \oplus fR \cong C$ for every idempotent f of R and every right R-module C. We observe that $M = I = \{x \in R : R \oplus xR \lesssim R\}$.

So assume that $R^2 \oplus fR \cong R \oplus C$, where $f \in R$ is idempotent and C is a right R-module. Write $\bar{x}$ for x +M. Since R/M is unit-regular, by [2,Thm.4.14] we have $R/M \oplus \bar{f}(R/M) \cong C/CM$. By applying [2,Prop.2.19] we get decompositions $R \oplus fR = A_1 \oplus A_2$ and $C = C_1 \oplus C_2$ such that $A_1 \cong C_1$ and $A_2 = A_2M$, $C_2 = C_2M$. By [2,Thm.2.8] there exist decompositions $R = D_1 \oplus D_2$, $fR = E_1 \oplus E_2$ such that $D_1 \oplus E_1 \cong A_1$ and $D_2 \oplus E_2 \cong A_2$. Observe that $D_2, E_2 \leqslant M$. By hypothesis we have $D_1 \cong R/D_2 \cong R$ and so we obtain

$$
\begin{aligned}
R \oplus fR &= D_1 \oplus D_2 \oplus E_1 \oplus E_2 \\
&\cong R \oplus D_2 \oplus E_1 \oplus E_2 \\
&\cong R \oplus C_2 \oplus D_2 \oplus E_1 \oplus E_2 \quad (\text{since } C_2 = C_2M) \\
&\cong R \oplus E_1 \oplus C_2 \qquad (\text{since } D_2 \oplus E_2 = (D_2 \oplus E_2)M) \\
&\cong D_1 \oplus E_1 \oplus C_2 \\
&\cong A_1 \oplus C_2 \\
&\cong C_1 \oplus C_2 = C .
\end{aligned}
$$

So $R \oplus fR \cong C$ as desired. □

LEMMA 4. *Let* R *be a right* $\aleph_0$*-continuous regular ring and set* $I = \{x \in R : R \oplus xR \lesssim R\}$. *If* $x \in I$ *then there exists an idempotent* $f \in I$ *with* $(fR)^2 \cong fR$ *such that* $xR \leqslant fR$.

Proof. By [1,Lemma 2.3] there exists an idempotent $g \in R$ such that $xR \lesssim gR$ and $gR \cong (gR)^2$. Let f be an idempotent of R such that $fR = xR + gR$. Clearly we have $gR \lesssim fR \lesssim gR$. By [1,Lemma 2.10] we have $gR \cong fR$. Therefore f is an idempotent such that $xR \leqslant fR$ and $fR \cong (fR)^2$. □

THEOREM 5. *Let* R *be a right* $\aleph_0$*-continuous regular ring and set* $I = \{x \in R : R \oplus xR \lesssim R\}$.

(i) If for every idempotent $e \in I$ *we have* $eR \lesssim (1 - e)R$, *then* R *is an Hermite ring.*

(ii) If there exists an idempotent $e \in I$ *such that* $eR \not\lesssim (1 - e)R$, *then the stable range of* R *is* ∞ .

Proof. (i) We shall see that I satisfies the conditions of Lemma 3. By [1,Theorem 2.7] R/I is unit-regular. Let $e \in I$ be an idempotent. By Lemma 4 there exists an idempotent $f \in I$ such that $eR \leqslant fR$ and $fR \cong (fR)^2$. By hypothesis, $fR \lesssim (1 - f)R$ and so there exists an idempotent $f' \in (1 - f)R(1 - f)$ such that $fR \cong f'R$. We have

$(1 - f)R = (1 - f - f')R \oplus f'R \cong (1 - f - f')R \oplus f'R \oplus fR \cong R$.

Consequently, $(1 - f)R \cong R$ and thus $R \cong (1 - f)R \lesssim (1 - e)R$.

By applying [1, Lemma 2.10], we obtain $(1 - e)R \cong R$. Hence $(1 - e)R \cong R$ for each idempotent $e \in I$. Now the result follows by Lemma 3.

(ii) Assume that $sr(R) \leqslant n$ for some $n \geqslant 2$, and let $e \in I$ be an idempotent such that $eR \not\lesssim (1 - e)R$. Since $e \in I$ we have

$$R^n \oplus (eR)^{n-1} \cong R^n \cong R \oplus ((1 - e)R)^{n-1} .$$

Hence by applying Theorem 1, we get $(eR)^{n-1} \lesssim ((1 - e)R)^{n-1}$. Now by [1, Theorem 2.13], $eR \lesssim (1 - e)R$. This is a contradiction. Thus the stable range of R is ∞ . □

Now we are ready to establish the main result of this paper.

THEOREM 6. *If* R *is a right* $\aleph_0$*-continuous regular ring then the stable range of* R *is 1, 2, or* ∞ . *Moreover, we have*

(a) $sr(R) = 1$ *if and only if* R *is directly finite.*

(b) $sr(R) \leqslant 2$ *if and only if* R *is an Hermite ring.* □

Recall that if R is a regular right self-injective ring then $R = R_1 \times R_2$ where R_1 is unit-regular and $R_2 \cong (R_2)^2$ as right R_2-modules. So, $sr(R) = \infty$ iff $R_2 \neq 0$. More generally one can prove that if R is a regular ring satisfying general comparability [2], then $sr(R) = \infty$ iff there exists a nonzero central idempotent $h \in R$ such that $(hR)^2 \lesssim hR$. However the analogous result for right $\aleph_0$-continuous regular rings fails. In fact there exists a prime, regular, right $\aleph_0$-continuous ring R such that

(i) $sr(R) = \infty$.

(ii) $R^n \lesssim R^m$ implies $n \leqslant m$ for all $n, m \geqslant 1$.

This ring can be constructed by introducing suitable modifications in an example of Goodearl [2,Example 14.35]. Let us adopt the notation of that example and introduce the following changes: take for $\mathcal{M}$ the class of all rings of the form $\Pi \mathrm{End}_F(V_i)$ where F is a fixed field and V_i are F-vector spaces, and take $S_1 = \mathrm{End}_F(V_1) \times \mathrm{End}_F(V_2)$ where $1 \leq \dim_F(V_1) < \aleph_0$ and $\dim_F(V_2) = \aleph_0$. We then obtain a ring R which is prime, regular, and right $\aleph_0$-continuous. Also, by construction, R has a factor ring isomorphic to S_1. It follows that $sr(R) = \infty$, and $R^n \lesssim R^m$ implies $n \leq m$ for all $n,m \geq 1$.

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