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INTRODUCTION

1. This volume is mainly concerned with the Nevanlinna factorization in classes of functions analytic in the unit disc $\mathbb{D}$ and smooth in a sense up to the boundary $\partial\mathbb{D}$ (in what follows we call such functions smooth analytic functions due to the smoothness of their boundary values). Different kinds of factorizations (i.e. roughly speaking methods of decomposition of a function into the "simplest" factors) played an important role in complex analysis from the very beginning of its existence and even now continue to be a keystone of that branch of mathematics. The Weierstrass products in the theory of entire functions, the Blaschke products, the inner and outer functions form nowadays an essential part of the analytic machinery . The last three decades have provided new inventions in that field - we would like mention only a long series of papers by M.M.Dzhrbashian [40] , a new factorization of entire functions introduced by Rubel [42] , and the Horowitz products [43] . Interest in the different kinds of factorizations is stimulated by urgent problems of Complex analysis and first of all by the problems of uniqueness and of the distribution of values, which form the core of the subject.

Factorizations as a tool are widely used in the study of ideals in Banach algebras of analytic functions, in problems of spectral analysis and synthesis; their vector-and operator-valued analogues play a notable role in the modern spectral operator theory.

In the present volume we deal with the least known and most frequently used factorization, namely with the Nevanlinna factorization or, in modern terms, the inner outer factorization. Developed by R.Nevanlinna, G.Szegö and V.I.Smirnov, the factorization was intensively studied already in the 1920-s and 1930-s. Nevertheless the development of methematics during the last few decades has revealed an essentially new phenomenon which roughly speaking consists in the fact that the Nevanlinna factorization fits well not only to classes similar to the Hardy classes but also to classes of smooth analytic functions.

Let us recall some classical facts and notation (see [44] , [45] for details).

2. A function I analytic and bounded in $\mathbb{D}$ is called an inner function if $\lim\limits_{r\to 1-0}|I(r\zeta)|=1$ almost everywhere on $\partial\mathbb{D}$. Two important examples of inner functions are the following:

a) Let $\{\alpha_K\}$ be a sequence (perhaps finite) of points of $\mathbb{D}\setminus\{0\}$ satisfying

$$\sum_K (1-|\alpha_K|)<\infty .$$

Then the product

$$B=\prod_K \frac{\bar{\alpha}_K}{|\alpha_K|}\frac{\alpha_K-z}{1-\bar{\alpha}_K z}$$

converges in $\mathbb{D}$ to an inner function vanishing at the α_K's and only at them. Then the function $z^m B$, where $m\in\mathbb{Z}_+$, is called a Blaschke product.

b) Let μ be a nonnegative Borel measure on the circle $\partial\mathbb{B}$ which is singular with respect to Lebesgue measure on $\partial\mathbb{D}$. The function

$$S_\mu(\tau)=\exp\Big(-\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{\zeta+\tau}{\zeta-\tau}\,d\mu(\zeta)\Big),\quad \tau\in\mathbb{D} \tag{0}$$

is an inner function. It does not vanish in $\mathbb{D}$ and is called the singular inner function corresponding to the measure μ .

These two examples are the basic ones because any inner function may be uniquely factored into the product

$$I=cBS \tag{1}$$

where $c \in \partial\mathbb{D}$, B is a Blaschke product and S is a singular function.

In what follows it is important to notice that as a rule I does not possess any smoothness on the circle $\partial\mathbb{D}$. In case I is continuous in $\bar{\mathbb{D}}$ we have $S \equiv 1$ and B is a finite Blaschke product.

c) Outer functions. Let $\log|h| \in L^1(\partial\mathbb{D})$. We can associate with h an analytic function ${}_e h$ in $\mathbb{D}$ (which is called the outer function corresponding the function $|h|$) as follows

$$
{}_e h(\tau) = \exp\left(\frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{e^{i\theta}+\tau}{e^{i\theta}-\tau} \log|h(e^{i\theta})|\,d\theta\right), \quad \tau \in \mathbb{D}.
$$

The function ${}_e h$ does not vanish in $\mathbb{D}$ and

$$
|{}_e h(\zeta)| = \lim_{r \to 1-0} |{}_e h(r\zeta)| = |h(\zeta)|
$$

for almost all $\zeta \in \partial\mathbb{D}$.

d) Class N. A function f analytic in $\mathbb{D}$ is said to belong to the Nevanlinna class N if

$$
\sup_{0<r<1} \int_0^{2\pi} \log^+ |f(re^{i\theta})|\,d\theta < \infty .
$$

That class plays a very important role in analysis. Classes of analytic functions most frequently used in harmonic analysis and operator-theory are usually contained in N.

The following result is the starting point of the Nevanlinna's factorization theory.

THEOREM. Let B be a Blaschke product, S_μ be a singular function corresponding to a real Borel measure μ and ${}_e h$ be an outer function corresponding to the function h such that $\log|h| \in L^1$.

Then

$$f = cBS_{\mu e} h \in N \tag{2}$$

Conversely, an arbitrary function $f \in N$ can be uniquely represented in the form (2). In what follows we write down the product in (2) as

$$f = {}_e f \cdot I_f$$

where $I_f = cBS_\mu$, ${}_e f = {}_e h$.

The importance of the Nevanlinna factorization is that it provides a complete description of the class as well as its crucial subclasses in terms of "pure real" parameters determining the factors I_f and ${}_e f$ (these are the constant $c(f) \in \partial\mathbb{D}$, the real measure $\mu = \mu(f)$), the sequence $\{\alpha_k\}$ and the number m and finally the values $|f|$ on $\partial\mathbb{D}$). The subclasses mentioned above are the Hardy classes H^p and the Smirnov class $\mathfrak{D}$. We recall [44] that

$$H^p = \{f \in N : \sup_{0<r<1} \int_0^{2\pi} |f(re^{i\theta})|^p d\theta < \infty\}, \quad 0<p<\infty,$$

$$H^\infty = \{f \in N : \sup_{\mathbb{D}} |f| < \infty\},$$

$$D = \{f \in N : \lim_{r\to 1-0} \int_0^{2\pi} \log^+ |f(re^{i\theta})| d\theta = \int_0^{2\pi} \lim_{r \to 1-0} \log^+ |f(re^{i\theta})| d\theta\}.$$

It can easily be cheked that $H^{p'} \subset H^{p''} \subset \mathfrak{D}$, if $\infty \geqslant p' \geqslant p'' > 0$. An alternative description of $\mathfrak{D}$ is the following: a function f belongs to $\mathfrak{D}$ iff the singular measure $\mu(f)$ is nonnegative. Thus the Hardy classes can be characterized as follows:

$$H^p = \{f \in N : \mu(f) \geqslant 0, \; f|_{\partial\mathbb{D}} \in L^p(\partial\mathbb{D})\}.$$

3. Let us now dwell on some details connected with the formula (2) which are especially important for our paper.

I. We say that an inner function I_2 divides an inner function I_1 if $I_1/I_2 \in H^\infty$. The above references yield that if $f \in \mathcal{D}$ and if an inner function I divides I_f then $fI^{-1} \in \mathcal{D}$. Similarly if $f \in H^p$ and I divides I_f then $fI^{-1} \in H^p$. Roughly speaking the function fI_f^{-1} is obtained from f by "removing the zeros" of f. As a matter of fact the function fI^{-1} does not vanish in $\mathbb{D}$, since fB_f^{-1} has no zeros in $\mathbb{D}$ and removing of f_μ means (in a sense) an isolation of the "boundary zeros" of f. The outer factor f behaves in Approximation Theory and Theory of Invariant Subspaces in many respects as invertible. Thus $\mathcal{D}$ and H^p are invariant with respect to the "isolation of zeros".

II. Nevanlinna's theorem contains the complete information about moduli of functions from N or H^p on $\partial\mathbb{D}$.

For example, let h be a nonnegative function on $\partial\mathcal{D}$. Then the following statements are equivalent:

(α) $\log h \in L^1(\partial\mathbb{D})$

(β) there exists an $f \in N$, $f \not\equiv 0$ such that

$$|f(\zeta)| = \lim_{r \to 1-0} |f(r\zeta)| = h(\zeta) \qquad \text{a.e. } \zeta \in \partial\mathbb{D} \tag{3}$$

We also have the equivalent statements (α_p) and (β_p):

(α_p) $\log h \,G\, L^1(\partial\mathbb{D})$, $h \in L^p(\partial\mathbb{D})$

(β_p) there exist an $f \in H^p$, $f \not\equiv 0$ such that (3) holds.

The inclusion $h \in L^p(\partial\mathbb{D})$ implies $\int_{\partial\mathbb{D}} \log h \, d\theta < \infty$, hence the statement ($\alpha_p$) may be rewritten in the form

(α_p') $\int_{\partial\mathbb{D}} \log h \, d\theta > -\infty$, $h \in L^p(\partial\mathbb{D})$.

Therefore the equivalence of (α'_p) and (β_p) yields a uniqueness theorem useful in applications,

$$f \in H^p, \quad \int_{\partial\mathbb{D}} \log|f|\,d\theta = -\infty \Rightarrow f \equiv 0 \tag{4}$$

III. Nevanlinna's theorem also contains a full description of the zero-sets of functions N : if $\{\alpha_K\}$ is a countable set in $\mathbb{D}$ then the following are equivalent:

(γ) there exists a function $f \not\equiv 0$, $f \in N$, such that $f^{-1}(0) = \{\alpha_K\}$

(δ) $\sum (1 - |\alpha_K|) < \infty$

IV. Nevanlinna's factorization is multiplicative:

$$e^{(fg)} = e^f \cdot e^g, \quad I_{fg} = I_f \cdot I_g .$$

These relations form an analytic basis of many important theorems concerning the structure of ideals or invariant subspaces in some spaces of analytic functions. A well-known (but not the only one) example is given by the famous Beurling theorem on the shift operator $f \mapsto zf$ on H^2.

4. We are now able to state (in a general form) four problems which are treated in the present notes.

I. What are the classes $X \subset \mathfrak{D}$ which are invariant with respect to the "isolation of the zeros" ?

II. What are the moduli $|f|_{\partial\mathbb{D}}$ of a given class $X \subset \mathfrak{D}$?

III. What are the zero sets of functions of a given class X ?

IV. What is the structure of closed ideals X if X is a Banach algebra (or what is the structure of shift invariant subspaces if X is a Banach space) ?

We postpone a detailed discussion and now only stress that we are going to study usual smooth analytic functions.

DEFINITION. Following V.P.Havin [4] we say that a class $X \subset \mathfrak{D}$

possesses the $(\mathfrak{F})$-property if for any $f \in X$ and for any inner function I dividing I_f the function fI^{-1} belongs to X.

We have already seen that H^p and $\mathfrak{D}$ possess the $(\mathfrak{F})$-property. That is a simple consequence of the factorization theorem. Because $f/I \in \mathfrak{D}$ and $|f/I|/\partial\mathbb{D} = |f|_{\partial\mathbb{D}}$ a.e. and the classes H^p are defined only in terms of $|f|_{\partial\mathbb{D}}$. But the statement that the disc-algebra C_A i.e. $f \in \mathfrak{D}$ (f is continuous in $\overline{\mathbb{D}}$)possesses the $(\mathfrak{F})$-property is deeper (this result was first stated by W.Rudin [5] in connection with his investigation of closed ideals in C_A). The disc-algebra C_A in contradiction to H^p and $\mathfrak{D}$ contains inner functions only as exception. Hence the $(\mathfrak{F})$-property in C_A is due to specific interference of outer and inner factors.

Much deeper than in C_A is an unexpected result of L.Carleson [3] , who has discovered the $(\mathfrak{F})$-property in the class

$$W^2_{1A} = \{f \in \mathfrak{D} : \iint |f'(x+iy)|^2 dx\, dy < \infty\}.$$

Moreover L.Carleson has succeeded in describing all of the parameters $\mu(f), \{\alpha_K\}, |f|_{\partial\mathbb{D}}$ of the factorization (2). Functions in W^2_{1A} (analytic functions with finite Dirichlet integral) have appropriate smoothness on $\partial\mathbb{D}$ and that numbers the investigation of the interplay of outer and inner factors.

B.I.Korenblum [6] , [7] has shown that the classes $H^2_n = \{f : f^{(n)} \in H^2\}$ possess the (F)-property. Such functions are already really smooth up to the boundary. Using a development of the method of [6] , [7] . V.P.Havin [4] has proved the same in the classes

$$H^p_n = \{f \in H^p : f^{(n)} \in H^p\},\quad n \geq 1,\quad 1 < p < \infty$$

and

$$\Lambda^{n+\alpha} = \{f : f^{(n)} \in \Lambda^\alpha\}$$

$$\Lambda^{\alpha} = \{ f : |f(z) - f(\zeta)| \leq C_f |z - \zeta|^{\alpha}, \quad z, \zeta \in \mathbb{D} \}, \quad 0 < \alpha < 1.$$

Independently and at the some time the (F)-property for $\Lambda^{n+\alpha}$ and some other classes was stated by F.A. Shamoyan.

The method mentioned above in some situations permits one to avoid an ingenious analysis of outer and inner factors. The main ideas are the following.

We define a Toeplitz operator $T_{\bar a}$:

$$(T_{\bar a} f)(\tau) = \frac{1}{2\pi i} \int_{\partial \mathbb{D}} \frac{\bar a(\zeta) f(\zeta)}{\zeta - \tau} d\zeta, \quad f \in X, \quad a \in H^{\infty}.$$

If we suppose that

$$T_{\bar a} X \subset X \qquad \text{for any} \quad a \in H^{\infty} \tag{5}$$

then X possesses the $(\mathcal{F})$-property. Indeed, if $a = I$, I is an inner function, I divides I_f then $T_{\bar I} f = f I^{-1}$ by the Cauchy formula. Following V.P.Havin [4] we call the property (5) of a class X the (K)-property. In [6a], [7], [4], [41] the $(\mathcal{F})$-property follows from the (K)-property of the corresponding class. The same implication was obtained in [39] by E.M.Dyn'kin in a different way. J.P.Kahane [46] has applied the best polynomial appronimation and has obtained the (F)-property in Λ^{α}, $0 < \alpha < 1$.

Taking in consideration all these results it may look quite natural that all "natural" classes possess the $(\mathcal{F})$-property.

However, it turned out that the most natural class

$$C_A^n = \{ f \in C_A : f^{(n)} \in C_A \}$$

is the most difficult one for the proof of the (F)-property.

It is not hard to prove that these classes do not possess the (K)-property. So it seems that the proof of the (F)-property must rest on a careful analysis of I_f and e_f.

The first paper, in which the $(\mathcal{F})$-property was studied directly (without use of Toeplitz operators) was that by S.A.Vinogradov and the author [49] (excluding the pioneer work of L.Carleson [3] where studying of the (F)-property was not the main purpose). It was shown in [49] that the space

$$H_1^1=\{f\in C_A: f^1\in H^1\}$$

possesses the (F)-property and the space

$$H_1^\infty=\{f\in C_A: f'\in H^\infty\}$$

"almost possesses" the (F)-property. For C_A^n and $H_n^\infty=\{f: f^{(n)}\in \in H^\infty\}$ the problem discussed was solved by the author [51], [53] with the help of a new method which permitted one to study in detail the rate of vanishing of $|f|$ in the vicinity of the critical set $\operatorname{spec} I = B^{-1}(0)\cup \operatorname{supp}\mu_s$, $I=BS$, I divides I_f. In the present volume we apply that method to classes of analytic functions with "varying boundary smoothness" (Ch. 1). The main result of § 1 is the following.

The class $\Lambda_\omega^n(\Phi)$ is a natural generalization of Λ_ω^n and is defined as follows:

$$\Lambda_\omega^n(\Phi)=\{f\in C_A: |f^{(n)}(z)-f^{(n)}(\zeta)|\leqslant$$

$$\leqslant c_\omega\omega(|\Phi((1-\frac{|z-\zeta|}{2})z)|\cdot|z-\zeta|),\quad z,\zeta\in\bar{\mathbb{D}}\}.$$

Φ is an outer function in $\mathbb{D}$ such that $\|\Phi\|_{\partial\mathbb{D}}\in A_1$, A_1 is the Muckenhoupt class, i.e. the class of nonnegative weights which for any are $I\subset\partial\mathbb{D}$ satisfy

$$\int_I|\Phi|\leqslant c|I|\operatorname*{ess\,inf}_I|\Phi|.$$

THEOREM 1. Let $n\geqslant 0$, ω be an arbitrary modulus of continuity and $|\Phi|_{\partial\mathbb{D}}\in A_1$. Then the class $\Lambda_\omega^n(\Phi)$ possesses the (F)-property.

It is natural to ask whether the multiplication of a function by its own inner factor retains the function in the considered class of smooth analytic functions as the division does.

If $f \in H^p$ and I is an inner function then obviously $fI \in H^p$. If $f \in C_A$, then fI in general does not belong to C_A but if we know in addition that I divides I_f then it is not difficult to check that $fI \in C_A$. The latter means the following. If I is an inner function then the conditions $f/I \in C_A$ and $fI \in C_A$ are equivalent. However this situation does not occur in classes of smooth analytic functions what is shown by Theorems 2 and 3.

THEOREM 2. Let $f \in \Lambda^n_\omega(\Phi)$, Φ be as in Theorem 1, I be an inner function, $f/I \in C_A$. Suppose that the multiplicity of the zeros of f at the points $\alpha \in \operatorname{spec} I \cap \mathbb{D}$ is at least $n+1$. Then $fI \in \Lambda^n_\omega(\Phi)$.

THEOREM 3. For any modulus of continuity ω there exists a function $f \in A^\infty$, $A^\infty = \bigcap_{n=1}^{\infty} C^n_A$ and a Blaschke product B such that $f/B \in A^\infty$, $fB \notin \Lambda^1_\omega$.

In spite of the abundance of examples of spaces of analytic functions with $(\mathcal{F})$-property, this property is not universal.

The first example of a space without $(\mathcal{F})$-property was pointed out by V.P.Gyrarii [9], who proved that the $(\mathcal{F})$-property is violated in

$$\ell^1_A = \{f \in C_A : \sum_{n \geq 0} |\hat{f}(n)| < \infty\}.$$

Later other examples were discovered:

$$\ell^p_A = \{f \in C_A : \sum_{n \geq 0} |\hat{f}(n)|^p < \infty\}$$

(for $p \in (1, \frac{4}{3})$ see [50] and for $p \in [\frac{4}{3}, 2]$ see [10]),

$$B_0 = \{f \in H^\infty : |f'(z)| = o((1-|z|)^{-1})\}.$$

(J.M.Anderson [11]).

In § 3 of Ch. 1 we exhibit new examples of classes without $(\mathfrak{F})$ - property.

THEOREM 4. Let $\{b_n\}$ be any sequence satisfying $1 \leqslant b_n \leqslant c_1 n^{c_2}$. Then the class

$$\{f \in H^1 : \sum_{n \geqslant 0} b_n |\hat{f}(n)|^p < \infty\}$$

does not possess the $(\mathfrak{F})$ -property for $p \in [1, \infty]$, $p \neq 2$.

II . Boundary values of the moduli of smooth analytic functions. Suppose first that the boundary values of a function $f \in \mathfrak{D}$ have some smoothness (for example $f \in Lip\,\alpha$). What then can be said about $|f|\,|\partial\mathbb{D}$? It is clear that $\log|f|$ must be summable on $\partial\mathbb{D}$ and that $|f|\,|\partial\mathbb{D} \in Lip\,\alpha$ $0 \leqslant \alpha < 1$. That is the only thing which is seen at the first glance But there exist much deeper observations. In [12] V.P.Havin and F.A.Shamoyan proved that for a nonnegative function h, $h \in Lip\,\alpha$, $0 < \alpha < 1$, $\log h \in L^1(\partial\mathbb{D})$, the outer function f with $|f|\,|\partial\mathbb{D} = h$ satisfies $f \in \Lambda^{\alpha/2}$. This result cannot be improved. There is a function $h \in Lip\,\alpha$ such that for the corresponding outer function f we have $f \in \Lambda^{\alpha/2+\varepsilon}$ for every $\varepsilon > 0$ (it is mentioned in [13] that a close result is contained in an unpublished paper by Jacobs). Later the theorem mentioned was generalized by V.P. Havin [13] to the class $Lip\,\omega$ for an arbitrary ω . The results of [12] and [13] pointed out that the boundary smoothness of an outer function must be half that of its modulus, whatever the understanding of the word "smoothness". In connection with his research in Approximation Theory J.Brennan [14] was forced to prove the theorem discussed for $h \in Lip\,\alpha$, $0 < \alpha < 2$. It is also worth mentioning that the implication $h \in Lip\,\alpha \Rightarrow {}_e h \in \Lambda^{\alpha/2}$ for any $\alpha > 0$ was used without proof as a crucial tool in papers by Taylor and Williams [15] and by Bruna and Ortega [47] . These authors referred to an unpublished paper by Carleson and Jacobs.

On the other hand in [51] a necessary and sufficient condition for the inclusion ${}_e h \in \Lambda^\alpha$ was found under assumption $h \in \mathrm{Lip}\,\alpha$. However, the form that condition was stated was not convenient for further generalizations.

In Ch. 2 we state a general result in that direction which concerns the scales Λ^α, α is not integer, H^p_n, $1<p<\infty$, $n \geq 1$ and $\Lambda^{n-1}Z$, Z is Zygmund class. Theorems 5, 6 and 7 correspond the classes cited. There is one idea of the description which can be realized in different ways depending of the situation. We shall use the common notation

$$L^p_n = \{f : \left(\frac{d}{d\theta}\right)^n f(e^{i\theta}) \in L^p(\partial\mathbb{D})\},$$

$\mathrm{Lip}\,\alpha$ for α not integer and

$$C^{n-1}Z = \{f : \left(\frac{d}{d\theta}\right)^{n-1} f(e^{i\theta}) \in Z(\partial\mathbb{D})\}.$$

We also introduce a specific notation: for a continuous function we put

$$M_h(z) = \max_{\substack{\zeta \in \partial\mathbb{D} \\ |\zeta - \frac{z}{|z|}| \leq 1-|z|}} |h(\zeta)|, \quad z \in \mathbb{D}, |z| \geq \frac{1}{2}.$$

Now we present a generalized statement of Theorems 5 - 7.

Let X be Λ^α, H^p_n or $\Lambda^{n-1}Z$ and Y be respectively $\mathrm{Lip}\,\alpha$, L^p_n or $C^{n-1}Z$. We put

$$H(X,\varphi,z) = \begin{cases} (1-|z|)^\alpha & \text{if } X = \Lambda^\alpha \\ (1-|z|)^n & \text{if } X = \Lambda^{n-1}Z \\ \varphi\left(\frac{z}{|z|}\right)(1-|z|^n) & \text{if } X = H^p_n. \end{cases}$$

(We emphasize that $H(X,\varphi,z)$ really depends on φ only in the case $X=H^p_n$)

a) Suppose that $f\in X$, $f\not\equiv 0$. For the suitable choice of φ we have the inequality

$$\int\limits_{\partial\mathbb{D}}\left|\log\left|\frac{M_f(z)}{f(\zeta)}\right|\right|\frac{1-|z|^2}{|\zeta-z|^2}\,|d\zeta|\leqslant C \tag{6}$$

which holds at every point $z\in\mathbb{D}\setminus\{0\}$ such that

$$M_f(z)\geqslant H(X,\varphi,z) \tag{7}$$

(C does not depend on z)

b) Suppose that $f\in Y$ and suppose that (6) holds with suitable choice of φ at any point $z\in\mathbb{D}\setminus\{0\}$ satisfying (7). Then $_e f\in X$

The boundedness of the integral (6) turns out to be quite a useful tool which permits one to investigate the behaviour of an analytic function rather carefully. Some corollaries of Theorems 5 - 7 are collected in Theorem 8.

THEOREM 8. Let $\int\limits_{\partial\mathbb{D}}\log|f|\,d\theta>-\infty$. Then

$$f\in Lip\,\alpha\Rightarrow_e f\in\Lambda^{\alpha/2} \tag{8}$$

$$f\in L^p_{2n},\ 1<p<\infty\Rightarrow_e f\in H^p_n \tag{9}$$

$$f_1,f_2\in\Lambda^\alpha,\ 0<\alpha<1,\ h(\zeta)=|f_1(\zeta)|+|f_2(\zeta)|\Rightarrow_e h\in\Lambda^\alpha.$$

The implication (8) strengthens the Carleson-Jacobs result cited by Taylor, Williams, Bruna and Orteya because we do not demand that f be nonnegative (in our case f can be a complex valued function) : F.A.Shamoyan [48] has proved the implication (9) for $f\geqslant 0$. Both (8) and (9) are ε -strict in the natural sense.

III. Local and global properties of zero-sets of smooth analytic