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James F. Peters · Andrzej Skowron

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Ivo Düntsch   Jerzy Grzymała-Busse
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Transactions on Rough Sets VI

Commemorating the Life and Work of Zdzisław Pawlak, Part I

Springer

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Preface

Volume VI of the *Transactions on Rough Sets* (TRS) commemorates the life and work of Zdzisław Pawlak (1926-2006)[1]. His legacy is rich and varied. Professor Pawlak's research contributions have had far-reaching implications inasmuch as his works are fundamental in establishing new perspectives for scientific research in a wide spectrum of fields.

From a very early age, Zdzisław Pawlak devoted his life to scientific research. The pioneering work by Prof. Pawlak included research on the design of computers, information retrieval, modeling conflict analysis and negotiation, genetic grammars, and molecular computing. His research led to the introduction of knowledge representation systems during the early 1970s and the discovery of rough sets during the early 1980s. Added to that was Prof. Pawlak's lifelong interest in painting, photography, and poetry. During his lifetime, he nurtured worldwide interest in approximation, approximate reasoning, and rough set theory and its applications[2]. Evidence of the influence of Prof. Pawlak's work can be seen in the growth in the rough-set literature that now includes over 4000 publications by more than 1600 authors in the rough set database[3] as well as the growth and maturity of the International Rough Set Society[4]. Numerous biographies of Zdzisław Pawlak have been published[5].

This volume of the TRS presents papers that reflect the profound influence of a number of research initiatives by Zdzisław Pawlak. In particular, this volume introduces a number of new advances in the foundations and applications of artificial intelligence, engineering, logic, mathematics, and science. These advances have significant implications in a number of research areas such as the foundations of rough sets, approximate reasoning, bioinformatics, computational intelligence, cognitive science, data mining, information systems, intelligent systems, machine intelligence, and security. In addition, it is evident from the papers included in this volume that rough set theory and its application form a very active research area worldwide. A total of 41 researchers from 8 countries are represented in this volume, namely, Canada, India, France, Norway, Poland, P.R.

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1 Prof. Pawlak passed away on 7 April 2006.

2 See, *e.g.*, Pawlak, Z., Skowron, A.: Rudiments of rough sets, *Information Sciences* 177 (2007) 3-27; Pawlak, Z., Skowron, A.: Rough sets: Some extensions, *Information Sciences* 177 (2007) 28-40; Pawlak, Z., Skowron, A.: Rough sets and Boolean reasoning, *Information Sciences* 177 (2007) 41-73.

3 `http://rsds.wsiz.rzeszow.pl/rsds.php`

4 `http://roughsets.home.pl/www/`

5 See, *e.g.*, Peters, J.F. and Skowron, A., Zdzisław Pawlak: Life and Work. *Transactions on Rough Sets* V, LNCS 4100 (2006) 1-24. See, also, R. Słowiński, Obituary, Prof. Zdzisław Pawlak (1926-2006), *Fuzzy Sets and Systems* 157 (2006) 2419-2422.

China, Sweden, Russia, Thailand, and the USA. Evidence of the vigor, breadth and depth of research in the theory and applications of rough sets can be found in the articles in this volume.

Most of the contributions of this commemorative volume of the TRS are on an invitational basis and every paper has been refereed in the usual way. This special issue of the TRS contains 23 papers and extended abstracts that explore a number of research streams that are either directly or indirectly related to research initiatives by Zdzisław Pawlak. These research streams are represented by papers on propositional logics (Mohua Banerjee and Md. Aquil Khan), intuitionistic rough sets for database applications (Theresa Beaubouef and Fred Petry), missing attribute value problem (Jerzy W. Grzymała-Busse and Witold J. Grzymała-Busse), Zdzisław Pawlak's contributions to the study of vagueness (Mihir Chakraborty), data mining (Alicja Wakulicz-Deja and Grzegorz Ilczuk), approximation of concepts (Anna Gomolińska), intelligent systems (Andrzej Jankowski and Andrzej Skowron), acoustics (Bozena Kostek), rule evaluation (Jiye Li, Puntip Pattaraintakorn, and Nick Cercone), rough sets in China (Qing Liu and Hui Sun), four-valued logic (Jan Małuszyński, Andrzej Szałas and Aida Vitória), crisp and fuzzy information systems (Alicja Mieszkowicz-Rolka and Leszek Rolka), artificial intelligence and rough sets (Tosiharu Munakata), topology and information systems (Piero Pagliani and Mihir K. Chakraborty), conjugate information systems (Maria Semeniuk-Polkowska), incomplete transactional databases (Grzegorz Protaziuk and Henryk Rybinski), classifiers, rule induction and rough sets (Jerzy Stefanowski), approximation spaces (Jarosław Stepaniuk), relevant attributes in high-dimensional data (Julio J. Valdés and Alan J. Barton), knowledge discovery in databases (Anita Wasilewska, Ernestina Menasalvas, Christelle Scharff), information quanta and approximation operators (Marcin Wolski), lattice theory for rough sets (Jouni Järvinen).

The editors of this volume extend their hearty thanks to reviewers of papers that have been submitted to the TRS during the past 12 months: Manuel Ojeda-Aciego, Mohua Banerjee, Jan Bazan, Mihir Chakraborty, Anna Gomolińska, Etienne Kerre, Pawan Lingras, Victor Marek, Piero Pagliani, Sheela Ramanna, Dominik Ślęzak, Jerzy Stefanowski, Jarosław Stepaniuk, Piotr Synak, Piotr Wasilewski and Yiyu Yao.

This issue of the TRS has been made possible thanks to the laudable efforts of a great many generous persons and organizations. The editors and authors of this volume also extend an expression of gratitude to Alfred Hofmann, Ursula Barth, Christine Günther and the LNCS staff at Springer for their support in making this volume of the TRS possible. In addition, the editors extend their thanks to Marcin Szczuka for his consummate skill and care in the compilation of this volume. The editors have been supported by the State Committee for Scientific Research of the Republic of Poland (KBN), research grant No. 3T11C00226, and the Natural Sciences and Engineering Research Council of Canada (NSERC) research grant 185986.

December 2006

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LNCS Transactions on Rough Sets

This journal subline has as its principal aim the fostering of professional exchanges between scientists and practitioners who are interested in the foundations and applications of rough sets. Topics include foundations and applications of rough sets as well as foundations and applications of hybrid methods combining rough sets with other approaches important for the development of intelligent systems.

The journal includes high-quality research articles accepted for publication on the basis of thorough peer reviews. Dissertations and monographs up to 250 pages that include new research results can also be considered as regular papers. Extended and revised versions of selected papers from conferences can also be included in regular or special issues of the journal.

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Table of Contents

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Monographs

Propositional Logics from Rough Set Theory

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Abstract. The article focusses on propositional logics with semantics based on rough sets. Many approaches to rough sets (including generalizations) have come to the fore since the inception of the theory, and resulted in different "rough logics" as well. The essential idea behind these logics is, quite naturally, to interpret well-formed formulae as rough sets in (generalized) approximation spaces. The syntax, in most cases, consists of modal operators along with the standard Boolean connectives, in order to reflect the concepts of lower and upper approximations. Non-Boolean operators make appearances in some cases too.

Information systems ("complete" and "incomplete") have always been the "practical" source for approximation spaces. Characterization theorems have established that a rough set semantics based on these "induced" spaces, is no different from the one mentioned above. We also outline some other logics related to rough sets, e.g. logics of information systems – which, in particular, feature expressions corresponding to attributes in their language. These systems address various issues, such as the temporal aspect of information, multiagent systems, rough relations.

An attempt is made here to place this gamut of work, spread over the last 20 years, in one platform. We present the various relationships that emerge and indicate questions that surface.

1 Introduction

A "logic of rough sets" would, in the natural sense, represent a formal system, statements in the language of which would be interpreted as rough sets in some approximation space. Thus "models" in the semantics of such a system would be approximation spaces, equipped with a meaning function that assigns rough sets to well-formed formulae (wffs) of the language.

Rough sets have been defined in more than one way for a Pawlak approximation space (X, R) – [1] lists five definitions, all of which are equivalent to each other. One of these is most commonly used:

(*) a rough set in (X, R), is the pair $(\underline{A}, \overline{A})$, for each $A \subseteq X$,

where $\underline{A}, \overline{A}$ denote the lower and upper approximations of A respectively. Another is a definition given by Pawlak in [2], and of interest to us in this paper:

* The author acknowledges the support of the Department of Computer Science, University of Regina, Canada, during a visit to which the paper was finalized.

J.F. Peters et al. (Eds.): Transactions on Rough Sets VI, LNCS 4374, pp. 1–25, 2007.

(**) $A \subseteq X$ is a rough set in (X, R), provided the boundary of A, $BnA \neq \emptyset$. For generality's sake, we could remove the restriction in (**) and consider definable sets (i.e. subsets with empty boundary) as special cases of rough sets.

Thus, in the semantics based on approximation spaces, the meaning function defining models, assigns to wffs either subsets of the domain, or pairs of subsets in accordance with (*) [3,4,5,6,7,8,9,10]. This is true even for semantics based on generalized approximation spaces, where different relations (may be more than one in number, with operations on them) are considered [6,11]. The logics invariably involve modalities to express the concepts of lower and upper approximations – some are simply known normal modal logics, or have non-Boolean connectives (and no modalities) in the language, but there are translations into modal logics. We make a study of this group of systems in Section 2. It may be remarked that the "rough logic" proposed by Pawlak [3] (the first system to be called so) makes an appearance here (cf. Section 2.6).

The "practical" source of Pawlak approximation spaces are *complete / deterministic* information systems. These have the form $\mathcal{S} \equiv (U, A, Val, f)$, where U is a set of objects, A a set of *attributes*, Val a set of *values* for the attributes, and f a function from $U \times A$ to Val. An equivalence relation $R_{\mathcal{S}}$ is induced on U (thus giving the approximation space $(U, R_{\mathcal{S}})$), as

$$x \; R_{\mathcal{S}} \; y \text{ in } U, \text{ if and only if } f(x, a) = f(y, a), \text{ for all } a \in A.$$

The converse also holds: given any approximation space (U, R), one can define an information system $\mathcal{S} \equiv (U, A, Val, f)$ such that the induced equivalence $R_{\mathcal{S}}$ is just the relation R. So, in effect, a semantics based on approximation spaces induced by complete information systems, is identical to the one discussed above.

Generalized information systems, termed *incomplete/nondeterministic*, are those where f is a function from $U \times A$ to $\mathcal{P}(Val)$, and yields different kinds of binary relations (e.g. similarity, inclusion – cf. Section 3.1) apart from equivalences, on U. Thus any information system (complete or incomplete) on a domain U, induces a relational system or a (generalized) approximation space on U, i.e. the (non-empty) set U together with a set of binary relations. This is called a *standard structure* on U [12,13,14]. For example, for the complete information system (U, A, Val, f) above, $(U, R_{\mathcal{S}})$ is a standard structure on U. In Section 3.1, $(U, sim_{\mathcal{S}}, in_{\mathcal{S}})$ is a standard structure for the incomplete information system $\mathcal{S} \equiv (U, A, Val, f)$, with similarity and inclusion relations $sim_{\mathcal{S}}, in_{\mathcal{S}}$. (Different sets of relations can give different standard structures on the same set U.)

The induced relations in the standard structure may be characterized by a set of properties. As we know, equivalences are characterized by the properties of reflexivity, symmetry and transitivity. The similarity and inclusion relations considered in Section 3.1 are characterized by the properties $(S1), (S2), (S4)$ – $(S6)$ given there. By a *general structure* on U [12,13,14], one means *any* relational system comprising a non-empty set, along with binary relations that satisfy the set of properties characterizing the induced relations in the standard structure. Again, for the complete information system (U, A, Val, f) above, any Pawlak approximation space (U, R) is a general structure. A general structure for $\mathcal{S}$ of

Section 3.1, would be of the form (U, sim, in), where sim, in are binary relations on U satisfying $(S1), (S2), (S4) - (S6)$.

One finds logics with semantics defined on incomplete information systems, for instance, in [15], or with semantics defined on general structures [16]. However, Vakarelov [12,13,14,17] has established a series of characterization results, enabling an identification of semantics based on general and standard structures (as in case of the Pawlak approximation space and complete information system above). In case of [15] too, we demonstrate here that the logic in question is equivalent to a normal modal logic with certain generalized approximation spaces defining models. These systems are discussed in Section 3.

In another line, there are "logics of information systems", which accommodate in their language, expressions corresponding to objects and attributes [18,19,4,20]. Amongst these is a system that addresses the temporal aspect of information (cf. [4]), while [20] presents a logic for multiagent systems. There are also treatises on "rough relations" – a logic has been proposed [21] on the one hand, and on the other, we have the proposal of a logic programming language in "rough datalog" [22]. In Section 4, we briefly sketch these and other approaches, such as rough mereology [23]. It will be seen that, some of the logics [4,16,20] have atomic propositions as (or built from) *descriptors*, the key feature of *decision logic* [2]. Decision logic is well-known, and not presented in this article.

One should mention that a few of the logics described here, have also been used as a base to express various concepts involving rough sets. For instance, Yao and Lin [6] have defined graded and probabilistic rough sets, using graded and probabilistic modal operators in the language of normal modal systems. Common and distributed knowledge operators have been interpreted in generalized approximation spaces by Wong [24]. In [25], another modal system (inspired by [3]) has been used to propose postulates for rough belief change.

A comparative study of the presented logics is made in Section 5. The paper concludes by indicating possible future directions of investigation in Section 6.

2 Logics with Semantics Based on Approximation Spaces

In this section, we take a look at logics with approximation spaces defining models. We find six kinds of systems.

For a logic $\mathcal{L}$, "α is a theorem of $\mathcal{L}$" shall be indicated by the notation $\vdash_{\mathcal{L}} \alpha$.

2.1 Normal Modal Systems

The modal nature of the lower and upper approximations of rough sets was evident from the start. Hence, it is no surprise that normal modal systems were focussed upon, during investigations on logics for rough sets. In particular, in case of Pawlak rough sets, the two approximations considered as operators clearly obey all the $S5$ laws. The formal connection between the syntax of $S5$ and its semantics in terms of rough sets is given as follows [26].

According to the Kripke semantics for $S5$, a wff α is interpreted by a function π as a subset in a non-empty domain U, the subset representing the extension of the formula – i.e. the collection of situations/objects/worlds where the wff holds. Moreover, in an $S5$-model $\mathcal{M} \equiv (U, R, v)$ (say), the accessibility relation R is an equivalence on U. Further, if $\Box$, $\Diamond$ denote the necessity and possibility operators respectively then for any wff α, $v(\Box\alpha) = \underline{v(\alpha)}$ and $v(\Diamond\alpha) = \overline{v(\alpha)}$.

A wff α is *true* in $\mathcal{M}$, if $v(\alpha) = U$. Now it can easily be seen that all the $S5$ theorems involving $\Box$ and $\Diamond$ translate into valid properties of lower and upper approximations.

Taking a cue from this connection, similar links have been pointed out (e.g. in [6,27]) between "rough sets" on generalized approximation spaces, and different normal modal systems. The basic idea is to define generalized approximation operators corresponding to any binary relation R on the domain U – this has been done by many (e.g. for tolerance relations in [28] and others – cf. [29]). More explicitly, a map $r : U \to \mathcal{P}(U)$ is defined as $r(x) \equiv \{y \in U : xRy\}$. Then the operators $\underline{apr}, \overline{apr} : \mathcal{P}(U) \to \mathcal{P}(U)$ are given by

$$\underline{apr}(A) \equiv \{x : r(x) \subseteq A\}, \text{ and } \overline{apr}(A) \equiv \{x : r(x) \cap A \neq \emptyset\}.$$

The rough set operators then satisfy various properties, depending upon the nature of R. Now let $\mathcal{L}$ denote a normal modal language, and $\mathcal{M} \equiv (U, R, v)$ be a model for $\mathcal{L}$. v, as before, interprets a wff as a subset in U. Then it is straightforward to observe that for any wff α of $\mathcal{L}$,

$$v(\Box\alpha) = \underline{apr}(v(\alpha)), \text{ and dually, } v(\Diamond\alpha) = \overline{apr}(v(\alpha)).$$

By the above interpretation, the modal logics like $KB, KT, K4, S5$ etc. could be said to capture the properties of rough sets in generalized approximation spaces based on different R (symmetric, reflexive, transitive, equivalence etc.).

As remarked in the Introduction, this link has been made use of further. Considering graded and probabilistic modal operators on the above systems, *graded* and *probabilistic* rough sets have been defined in [6]. Wong [24] has interpreted common and distributed knowledge operators (as defined in logic of knowledge) in generalized approximation spaces with an indexed set of indiscernibility relations (corresponding to the knowledge operator of each agent).

2.2 *DAL*

[11] considers generalized approximation spaces containing a family of equivalence relations instead of just one. The logic DAL that is defined in [11], has models based on these spaces. Further, the set of equivalence relations is assumed to be closed with respect to the operations of intersection and transitive closure of union of relations.

The language of DAL, expectedly, includes a family of modal operators intended to correspond to the indiscernibility relations on the domains of the models. Formally, this is done by having a set $\mathcal{R}$ (say) of *relational variables* apart from the set $\mathcal{P}$ of propositional ones. There are binary operations $\cap$, $\uplus$, and a collection REL of *relational expressions* is built inductively out of the members of $\mathcal{R}$ with these operations. Apart from the classical Boolean connectives, a modal connective $[R]$ is then introduced in the language for each $R \in REL$.

A DAL-model is a structure $\mathcal{U} \equiv (U, \{\rho_R\}_{R \in REL}, m)$, where, (i) for any $R \in REL, \rho_R$ is an equivalence relation in the set U; (ii) $\rho_{R \cap S}$ is the greatest equivalence relation in U included in both ρ_R and ρ_S; (iii) $\rho_{R \uplus S}$ is the least equivalence relation including both ρ_R and ρ_S; and (iv) m is the meaning function from $\mathcal{P} \cup \mathcal{R}$ to $\mathcal{P}(U) \cup \{\rho_R\}_{R \in REL}$ such that $m(p) \subseteq U$, for $p \in \mathcal{P}$, and $m(R) \equiv \rho_R$, for $R \in REL$.

For evaluating truth of wffs in DAL-models, one defines a function v that is determined by the meaning function m:

$$v(p) \equiv m(p), \text{ for } p \in \mathcal{P},$$

$$v([R]\alpha) \equiv \{x \in U : y \in v(\alpha), \text{ for all } y \text{ such that } x \; m(R) \; y\},$$

the Boolean cases being defined in the standard way.

Definitions of truth and validity then are as usual: α is true in $\mathcal{U}$, provided $v(\alpha) = U$, and valid if it is true in all DAL-models.

DAL has been axiomatized as follows. The connective $\langle\rangle$ is the dual of $[]$.

$A1$. All classical tautologies,

$A2$. $[R](\alpha \to \beta) \to ([R]\alpha \to [R]\beta)$,

$A3$. $[R]\alpha \to \alpha$,

$A5$. $\langle R\rangle\alpha \to [R]\langle R\rangle\alpha$,

$A5$. $[R \uplus S]\alpha \to [R]\alpha \wedge [S]\alpha$,

$A6$. $(([P]\alpha \to [R]\alpha) \wedge ([P]\alpha \to [S]\alpha)) \to ([P]\alpha \to [R \uplus S]\alpha)$,

$A7$. $[R]\alpha \vee [S]\alpha \to [R \wedge S]\alpha$,

$A8$. $(([R]\alpha \to [P]\alpha) \wedge ([S]\alpha \to [P]\alpha)) \to ([R \wedge S]\alpha \to [P]\alpha)$.

The only rules of inference are Modus Ponens and Necessitation (corresponding to the connective $[R]$ for each $R \in REL$).

The axiomatization yields a completeness result with respect to the aforementioned semantics.

Theorem 1. *For any DAL-wff α, $\vdash_{DAL} \alpha$, if and only if α is valid.*

2.3 Pre-rough Logic

Following in the footsteps of Rasiowa, the algebra of rough sets was investigated in [7] in order to arrive at a logic for the theory. An algebraic structure called *pre-rough algebra* was proposed – this is a *quasi Boolean algebra* [30] along with a topological operator satisfying all the properties of an *interior*, and more. A corresponding logic PRL was framed, and observed to be sound and complete with respect to a semantics based on rough sets.

The language of PRL has the primitive logical symbols $\neg, \sqcap, \Box$. $\sqcup, \Diamond$ are duals of $\sqcap, \Box$, while $\Rightarrow$ is defined as:

$$\alpha \Rightarrow \beta \equiv (\neg\Box\alpha \sqcup \Box\beta) \sqcap (\neg\Diamond\alpha \sqcup \Diamond\beta),$$

for any wffs α, β of PRL.