Lecture Notes in Mathematics 1511

B. S. Yadav D. Singh (Eds.)

Functional Analysis and Operator Theory

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Lecture Notes aim to report new developments - quickly, informally and at a high level. The following describes criteria and procedures for multi-author volumes. For convenience we refer throughout to "proceedings" irrespective of whether the papers were presented at a meeting.

The editors of a volume are strongly advised to inform contributors about these points at an early stage.

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Further remarks and relevant addresses at the back of this book.

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P R E F A C E

The department of mathematics of the University of Delhi South Campus, organised a National Seminar from August 2-4, 1990 and an International Conference from August 5-6, 1990 in memory of the late Professor U.N. Singh. The theme of the seminar as well as the conference was 'Recent Trends in Contemporary Analysis'. This volume comprises the proceedings of the conference and also includes papers by mathematicians who were unable to attend. The topics that are covered include Functional Analysis, Operator Theory, Abstract Harmonic Analysis, Fourier Analysis Approximation Theory and Function Theory.

The volume is dedicated to the memory of Professor U.N. Singh who was a distinguished analyst and amongst the pioneers who initiated the study of Functional Analysis in India. It includes a paper by him written a few weeks before his demise.

The department is grateful to the University Grants Commission of India and the National Board of Higher Mathematics for their financial support. We would like to express our gratitude to Professor P.K. Jain, Professor R. Vasudevan, Dr. Pramod Kumar, Dr. S.C. Arora and Dr. Ajay Kumar for their help in the preparation of this volume and to Savitri Devi, Poonam Satija and Preeti Nigam for their diligent proof reading.

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Finally, we would like to express our thanks to Springer Verlag for their interest and cooperation in bringing out this volume.

B.S. Yadav

Dinesh Singh

CONTENTS

A Qualitative Uncertainity Principle for Hypergroups

AJAY KUMAR

Dedicated to the memory of U.N. Singh

It is known that for a locally compact abelian group G if $f \in L^1(G)$ and the product of the measure of the support of f and its Fourier transform $\hat{f}$ is less than one then f = 0 a.e. It is also known that if G is with a noncompact identity component and the measure of the support of each f and its Fourier transform $\hat{f}$ is finite, then f = 0 a.e. In this paper we study generalizations of these results for commutative hypergroups.

The uncertainity principles in Fourier analysis assert that the more a function f is concentrated, the more its Fourier transform $\hat{f}$ will be spread out. It has been known for quite sometime that if $f \in L^2(\mathbb{R}^n)$ and the support of f and that of its Fourier transform $\hat{f}$ are bounded, then f = 0 a.e.

For a commutative hypergroup K equipped with Haar measure m, let $\hat{K}$ be the dual space with Plancherel measure π [6] and [7]. For $f \in L^1(K)$, let $A_f=\{x\in K: f(x)\neq 0\}$ and $B_f=\{\gamma\in\hat{K}: \hat{f}(\gamma)\neq 0\}$. The main aim of this paper is to show that if $m(A_f)\pi(B_f) < 1$, then f=0 a.e. If K has a noncompact identity component (with some additional continuity condition) and $m(A_f)<\infty$, $\pi(B_f)<\infty$, then f = 0 a.e.

For $K = \mathbb{R}^n$, the above results have been proved by Matolcsi and Szücs [8] and Benedicks [1] and for locally compact abelian groups by Hogan [5]. Some of the proofs of our results are largely inspired by [5] and [8]. Non abelian groups have been studied by Price and Sitaram [9], Cowling, Price and Sitaram [3] and Echterhoff, Kaniuth and Kumar [4].

Let B(K) denote the σ-algebra of Borel subsets of K and ξ_E the characteristic function of $E \in B(K)$. For $E \in B(K)$ and $F \in B(\hat{K})$, define projections

$P(E): L^2(K) \to L^2(K)$ by $f \to f\xi_E$ and $P^1(F) : L^2(\hat{K}) \to L^2(\hat{K})$ by $\phi \to \phi\xi_F$

Clearly $||P(E)|| \le 1$ and $||P^1(F)|| \le 1$. For $f \in L^2(K)$ and $F \in B(\hat{k})$, let Q be the projection on $L^2(K)$ defined by $Q(F)f = (P^1(F)\hat{f})^{\check{}}$. For projections P_1 and P_2 on a Hilbert space H denote by $P_1 \wedge P_2$ the projection on $P_1H \cap P_2H$.

THEOREM 1. If $E \in B(K)$ and $F \in B(\hat{K})$ are such that $m(E)\pi(F) < 1$, then $P(E) \wedge Q(F) = 0$.

PROOF. If $m(E) = 0.$ or $\pi(F) = 0$, then for $f \in L^2(K)$, $P(E)f = 0$ or $Q(F)f = 0$, so suppose that $m(E) \neq 0$ and $\pi(F) \neq 0$. Now $m(E)\pi(F) < 1 \Rightarrow m(E) < \infty$ and $\pi(F) < \infty$. Let $g = (\bar{\xi}_F)^{\check{}}$. By ([6], 12.1I), it follows that $g \in L^2(K)$. For any $x \in K$,

$$|g(x)| = |\int_K \gamma(x)\xi_F(\gamma)d\pi(\gamma)| \le \int_F |\gamma(x)|d\pi(\gamma) \le \pi(F).$$

So for any $x,y \in K$ we have $|g(x*y)| \le \pi(F) \int_K dp_x * p_y(z) = \pi(F)$.
Now for any $\psi \in L^1(K) \cap L^2(K)$ and $x \in K$,

$$(P(E)Q(F)\psi)(x) = \xi_E(x)\ (P^1(F)\hat{\psi}\ )^{\check{}}(x) = \xi_E(x)(\bar{\xi}_F\ \hat{\psi}\ )^{\check{}}(x)$$
$$= \xi_E(x)\ \bar{\xi}_F^{\check{}} * \psi(x) = \xi_E(x)\ g*\psi(x)$$

Thus $||P(E)Q(F)\psi||_1 \le m(E)\pi(F)\ ||\psi||_1$ and

$$||P(E)Q(F)\psi||_2 \le (m(E))^{1/2}\pi(F)||\psi||_1.$$

Therefore, $P(E)Q(F)\psi \in L^1(K) \cap L^2(K)$. Applying the above inequalities to $P(E)Q(F)\psi$, we get

$$||(P(E)Q(F))^2\psi||_1 \le (m(E))^2(\pi(F))^2 \quad ||\psi||_1$$

and $$||(P(E)Q(F))^2\psi||_2 \le (m(E))^{3/2}(\pi(F))^2||\psi||_1.$$

Thus by inducton

$$||(P(E)Q(F))^n\psi||_2 \le (m(E))^{n-1/2}(\pi(F))^n||\psi||_1.$$

Therefore $||(P(E)Q(F))^n\psi||_2 \to 0$ as $n \to \infty$.

Let $f \in L^2(K)$.There exists $f_1 \in L^1(K) \cap L^2(K)$ such that $||f-f_1|| < \epsilon/2$.

Choose n large enough so that $||(P(E)Q(F))^n f_1|| < \epsilon/2$. Hence

$$||(P(E)Q(F))^n f||_2 \le ||(P(E)Q(F))^n(f-f_1)||_2 + ||(P(E)Q(F))^n f_1||_2 < \epsilon.$$

So $||(P(E)Q(F))^n f||_2 \to 0$ as $n \to \infty$ for all $f \in L^2(K)$.

Thus for $f \in P(E)(L^2(K)) \cap Q(F)(L^2(K))$, we have

$$||f||_2 = ||(P(E)Q(F))^n f||_2 \to 0 \text{ so } f = 0.$$

COROLLARY 2. Let $f \in L^1(K)$ be such that $m(A_f)\pi(B_f) < 1$.Then $f = 0$ a.e.

PROOF. We first remark that $f \in L^1(K)$, $m(A_f) <\infty$ and $\pi(B_f) < \infty$ if and only if $f \in L^2(K)$, $m(A_f)<\infty$ and $\pi(B_f)<\infty$.Clearly $P(A_f)f= f$ and $Q(B_f)f = (P^1(B_f)\hat{f})^{\vee} = (\hat{f})^{\vee} = f$ by ([6], 12.2C). So $P(A_f)Q(B_f)f= f$. Hence $||f|| = ||(P(A_f)Q(B_f))^n f|| \to 0$, and hence $f = 0$ a.e.

Now we proceed to prove the following version of the qualitative uncertainity principle (QUP). K is said to satisfy QUP if for each $f \in L^1(K)$, $m(A_f)<\infty$ and $\pi(B_f)<\infty \Rightarrow f = 0$ a.e.

LEMMA 3. If C is a compact subset of K, then the map $a \to m(a * C)$ is continuous.

PROOF. For $a \in K$, $a*C$ is compact. By regularity of the Haar measure, we have for $\epsilon>0$, there exists an open set W such that $a*C \subset W$ and $m(W)<m(a*C)+\epsilon$. Now for every $c \in C_1 = a*C$ there exists a neighbourhood U_c of e such that $c * U_c \subset W$. Also there exists a neighbourhood V_c of e such that $V_c*V_c \subset U_c$. Now $\{ c*V_c : c \in C_1\}$ covers C_1, therefore, there exists a finite subcover

$\{ c_i*V_{c_i} : 1 \leq i \leq n\}$ of C_1. Let $V = \bigcap_{i=1}^{n} V_{c_i}$. Then

$$a*C*V \subset \bigcup_{i=1}^{n} c_i * V_{c_i} * V \subset \bigcup_{i=1}^{n} c_i * V_{c_i} * V_{c_i} \subset \bigcup_{i=1}^{n} c_i * U_{c_i} \subset W.$$

Thus $m(a*C*V) \leq m(W) < m(a*C)+\epsilon$. Hence by using ([6],3.3C) it follows that the map $a \to m(a*C)$ is continuous.

Using the above lemma, it follows easily that if K is a compact hypergroup, a discrete hypergroup or a locally compact group and $C \subseteq K$ is such that $0 < m(C) < \infty$, then the map $a \to m(a*C)$ is continuous. We don't know whether the above lemma can be extended to a subset C of K with $0<m(C)<\infty$. However, we assume in the remaining part that K satisfies the condition that $a \to m(a*C)$ is continuous for every $C \subseteq K$ with $0<m(C)<\infty$.

PROPOSITION 4. Let K be a commutative hypergroup with anoncompact identity component K_o. Let C be a measurable subset of K with $0<m(C)<\infty$. If $C_o \subseteq C (m(C_o)>0)$ and $\epsilon>0$, then there exists $a \in K_o$ such that

$$m(C) < \int_K (\xi_C + \xi_{a*C_o} - \xi_C \, _{\tilde{a}}(\xi_{C_o}))dm < m(C) + \epsilon.$$

PROOF. Define $h : K_o \to \mathbb{R}^+$ by $h(a) = \int_K (\xi_C + \xi_{a*C_o} - \xi_C\, {}_{\tilde{a}}(\xi_{C_o}))\, dm$ so that $h(a) = m(C) + m(a*C_o) - \langle {}_a(\xi_{C_o}), \xi_C \rangle$. By assumption and the continuity of the map $a \to {}_a f$ ([2], §2), it follows that h is a continuous function. Select $\delta > 0$ such that $0 < 2\delta < m(C_o)$. There exists a compact set $F \subseteq C$ such that $m(C \sim F) < \delta$. Let $M = \tilde{F} * F$, by ([6],3.2B) M is a compact subset of K. Since $e \in M$ and $e \in K_o$, $M \cap K_o$ is compact. As K_o is non-compact, select $a \in K_o \sim (M \cap K_o)$. It is easy to see that ${}_a(\xi_F)\xi_F = 0$. If $x \notin F$, then clearly ${}_a(\xi_F)\xi_F(x) = 0$. If $x \in F$, then

${}_a(\xi_F)\xi_F(x) = \xi_F(a*x) = 0$, since $\{a\}*\{x\} \cap F = \emptyset$. In fact, if $y \in \{a\}*\{x\} \cap F$, then $a \in \{\tilde{x}\}*\{y\} \subseteq \tilde{F} * F = M$, and hence $a \in M \cap K_o$, which is a contradiction. Next we claim that

$\int_K {}_{\tilde{a}}(\xi_{C_o})\, \xi_F(x)\, dm(x) < \delta$. In fact,

$$\int_K {}_{\tilde{a}}(\xi_{C_o})\, \xi_F(x)\, dm(x) =$$

$$\int_K {}_{\tilde{a}}(\xi_{C_o \cap F} + \xi_{C_o \cap F'})(x)\, \xi_F(x)\, dm(x)$$

$$= \int_K \xi_{C_o}(x)\, \xi_F(x)\, {}_a(\xi_F)(x)\, dm(x) + \int_K \xi_{C_o \cap F'}\, {}_a(\xi_F)\, dm \quad ([6],\ 5.1D)$$

$$= \int_K \xi_{C_o \cap F'}\, {}_a(\xi_F)\, dm \leq \int_K \xi_{C_o \cap F'}\, dm = m(C_o \cap F')$$

$$\leq m(C \sim F) < \delta.$$

Thus

$$h(a) = \int_K (\xi_C + \xi_{a*C_o} - \xi_C \tilde{_a}(\xi_{C_o}))(x)dm(x)$$

$$= m(C) + m(a*C_o) - \int_K {}_a(\xi_{C\cap F} + \xi_{C\cap F'})\, \xi_{C_o} dm \qquad ([6],5.1\ D)$$

$$\geq m(C) + m(C_o) - \int_K {}_a(\xi_F)\, \xi_{C_o} dm - \int_K \xi_{C\cap F'}\, \tilde{_a}(\xi_{C_o}) dm \quad (\xi_{a*C_o} \geq \tilde{_a}(\xi_{C_o}))$$

$$\geq m(C) + m(C_o) - \int \xi_F\, \tilde{_a}(\xi_{C_o}) dm - \int \xi_{C\cap F'}\, dm$$

$$> m(C) + m(C_o) - (m(C_o)/2) - (m(C_o)/2 = m(C) = h(e).$$

Hence h is a nonconstant continuous function on the connected set K_o and $h(a)>h(e)$. We may now choose $a_o \in K_o$ such that

$$m(C)=h(e) < h(a_o) = \int_K (\xi_C + \xi_{a*C_o} - \xi_C \tilde{_{a_o}}(\xi_{C_o}))(x)dm(x)$$

$$< h(e) + \epsilon = m(C) + \epsilon.$$

THEOREM 5. If K is a commutative hypergroup with a noncompact identity component such that the map $a \to m(a*C)$ is continuous for all $C \subseteq K$ with $o < m(C) < \infty$, then K satisfies QUP.

PROOF. Let $f \in L^1(K)$ be such that $m(A_f) < \infty$ and $\pi(B_f) < \infty$. Suppose $f_o \in P(A_f)(L^2(K)) \cap Q(B_f)(L^2(K))$ and $f_o \neq 0$. Let $A_o = \{x \in K : |f_o(x)| > 0\}$ so $m(A_o) > 0$. Select $N \in \mathbb{N}$ with $2m(A_o)\pi(B_f) < N$. Take $C_o = A_o$ and $C = A_o$ in the above proposition.

There exists $a_1 \in K_o$ such that

$$m(A_o) < \int_K (\xi_{A_o} + \xi_{a_1*A_o} - \xi_{A_o} \tilde{_{a_1}}(\xi_{A_o}))(x)dm(x) < m(A_o) + 1/2\pi(B_f).$$

Take $C_o=A_o$ and $C = A_o \cup (a_1 * A_o) = A_1$ in the above proposition. Then there exists an $a_2 \in K_o$ such that

$$m(A_1) < \int_K (\xi_{A_1} + \xi_{a_2 * A_o} - \xi_{A_1}\, \tilde{a_2}(\xi_{A_o}))dm < m(A_1) + 1/(2\pi(B_f))$$

Repeating the above process we get $A_i = A_{i-1} \cup (a_i * A_o)$ with $a_i \in K_o$ satisfying

$$m(A_{i-1}) < \int_K (\xi_{A_{i-1}} + \xi_{a_i * A_o} - \xi_{A_{i-1}} a_i(\xi_{A_o}))dm < m(A_{i-1}) + 1/(2\pi(B_f))$$

As in the proof of Theorem 1, $P(A_i)Q(B_f)\varphi(x) = \xi_{A_i}(x)(\xi_{B_f})^{\gamma} * \varphi(x)$.

Using ([5],1.2), it follows that

$$||P(A_1)Q(B_f)||_2^2 \le \int_K \int_K |\xi_{A_i}(x)(\xi_{B_f})^{\gamma}(x * \tilde{y})|^2\, dm(y)\, dm(x)$$

$$= \int_{A_i} \int_K |{}_x((\xi_{B_f})^{\gamma})(\tilde{y})|^2\, dm(y)dm(x)$$

$$= \int_{A_i} \int_{\hat{K}} |\gamma(x)\, \xi_{B_f}(\gamma)|^2\, d\pi(\gamma)dm(x)$$

$$\le m(A_i)\pi(B_f).$$

Thus $\dim\, (P(A_N)(L^2(K)) \cap Q(B_f)(L^2(K)) \le m(A_N)\pi(B_f)$

$$= m(A_{N-1} \cup (a_N * A_o)\pi(B_f)$$

$$= \int_K (\xi_{A_{N-1}} + \xi_{a_N * A_o} - \xi_{A_{N-1}}\, \xi_{a_N * A_o})dm\, \pi(B_f)$$

$$\le \int_K (\xi_{A_{N-1}} + \xi_{a_N * A_o} - \xi_{A_{N-1}}\, \tilde{a}_N(\xi_{A_o}))dm\, \pi(B_f)$$

$< (m(A_{N-1}) + 1/(2\pi(B_f))\pi(B_f)$

$\vdots$

$< (m(A_o) + N/(2\pi(B_f))\pi(B_f) \; < \; N/2 + N/2 \; = \; N.$ ------------ (1)

Let $f_i = {}_{a_i}(f_o)$ so that $\hat{f}_i(\gamma) = \gamma(a_i)\hat{f}_o(\gamma)$.

Now $Q(B_f)f_i(x) = \int_{\hat{K}} \xi_{B_f}(\gamma)\gamma(a_i * x)\, \hat{f}_o(\gamma)d\pi(\gamma)$

$= Q(B_f)f_o\,(a_i * x) = f_o(a_i * x) = f_i(x)$

for all $0 \leq i \leq N$.

As supp $f_i = \{a_i\}*$ supp f_o, $f_i = 0$ a.e. on $(a_i * A_o)$.

Since $P(A_m)f_i(x) = \xi_{A_m}(x)\; f_o(a_i * x)$, we have

$P(A_m \sim A_{m-1})\; f_m(x) = \xi_{A_m \sim A_{m-1}}(x)\; f_o(a_m * x) \neq 0$

and $P(A_m \sim A_{m-1})\; f_i(x) = 0$ for $0 \leq i \leq m-1$. Therefore f_m is not a linear combination of $f_o, f_1 \ldots f_{m-1}$ and hence $\{ f_o, f_1 \ldots f_N\}$ is a set of N+1 linearly independent set in $P(A_N)(L^2(K)) \cap Q(B_f)(L^2(K))$.

This leads to a contradiction to (1). Hence $f_o = 0$ a.e.

I would like to thank Prof.(Mrs.) Ajit I. Singh for some useful discussions.

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