Lecture Notes in Mathematics

Edited by A. Dold and B. Eckmann

1310

Olav Arnfinn Laudal
Gerhard Pfister

Local Moduli and Singularities

Springer-Verlag

Lecture Notes in Mathematics

Edited by A. Dold and B. Eckmann

1310

Olav Arnfinn Laudal
Gerhard Pfister

Local Moduli and Singularities

Springer-Verlag
Berlin Heidelberg New York London Paris Tokyo

Authors

Olav Arnfinn Laudal
Universitetet i Oslo, Matematisk Institutt
P.O. Box 1053 Blindern, 0316 Oslo 3, Norway

Gerhard Pfister
Humboldt-Universität zu Berlin, Sektion Mathematik
1080 Berlin, Unter den Linden 6, German Democratic Republic

Mathematics Subject Classification (1980): 14D15, 14D20, 14B07, 14B12, 16A58, 32G11

ISBN 3-540-19235-2 Springer-Verlag Berlin Heidelberg New York
ISBN 0-387-19235-2 Springer-Verlag New York Berlin Heidelberg

Printing and binding: Druckhaus Beltz, Hemsbach/Bergstr.
2146/3140-543210

NOTATIONS

k	field	
W^{*}	the dual of the k-vektor space W	
$k[\underline{x}] = k[x_1,\ldots,x_n]$		
$\underline{P}^n$	the projective n-space	
$\underline{m}_S$	the maximal ideal of a local ring S	
$S^{\wedge}$	the completion of the local ring S	
$\underline{\ell}$	the category of local artinian k-algebras with residue field k	
$\underline{\ell}_{H^{\wedge}}$	the category of local artinian $H^{\wedge}$-algebras with residue field k	
$\underline{gr}$	the category of groups	
$- \otimes - = - \otimes_k -$		
$\underline{R} = \operatorname{Spec} R$, R a commutative ring		
$\operatorname{Der}_k^c(A)$	the continuous derivations of a complete local k-algebra A	
Def_X	the deformation functor of X	§1
A^i	the generalized André cohomology (see [La 1])	§1,3
H^i	the André cohomology for algebras (see [An],[La 1])	§2
T^i	$\operatorname{Sym}_k(A^{i*})^{\wedge}$ (see [La 1])	§1
$\underline{H}^{\wedge}$	the formal moduli of X, i.e. the (prorepresenting) hull of Def_X	§1
$\underline{H}_o^{\wedge}$	the prorepresenting substratum of $\underline{H}^{\wedge}$	§1
$\underline{H}_{(n)}^{\wedge}$	the n-th equicohomological substratum of $\underline{H}^{\wedge}$	§1
$\pi^{\wedge} : \underline{X}^{\wedge} \to \underline{H}^{\wedge}$	the formal versal family	§1
$\operatorname{aut}_S(X\otimes S) = \{\phi \in \operatorname{Aut}_S(X\otimes S) \mid \phi\otimes_S k = 1\}$, S in $\underline{\ell}$		§1
$\underline{\operatorname{aut}}_X$	the group-functor defined by $\underline{\operatorname{aut}}_X(S) = \operatorname{aut}_S(X\otimes S)$	§1
$\operatorname{aut}_R(X^{\wedge}\otimes_{H^{\wedge}} R) = \{\phi \in \operatorname{Aut}_R(X^{\wedge}\otimes_{H^{\wedge}} R) \mid \phi\otimes_R k = 1\}$, R in $\underline{\ell}_{H^{\wedge}}$		§1
$\underline{\operatorname{aut}}_{X^{\wedge}}$	the group-functor defined by $\underline{\operatorname{aut}}_{X^{\wedge}}(R) = \operatorname{aut}_R(X^{\wedge}\otimes_{H^{\wedge}} R)$	§1
$\operatorname{aut}_R(H^{\wedge}\otimes R) = \{\phi \in \operatorname{Aut}_R(H^{\wedge}\otimes R) \mid \phi(\underline{m}_{H^{\wedge}}\otimes R) \subseteq \underline{m}_{H^{\wedge}}\otimes R\}$		§2
$\underline{\operatorname{aut}}_{H^{\wedge}}$	the group-functor defined by $\underline{\operatorname{aut}}_{H^{\wedge}}(R) = \operatorname{aut}_R(H^{\wedge}\otimes R)$	§1
$\underline{I}_{\pi^{\wedge}}$	the subgroup-functor of $\underline{\operatorname{aut}}_{H^{\wedge}}$ leaving $\pi^{\wedge}$ invariant	§2

$\underline{i}_{\pi}\wedge = \underline{i}_X$	the subgroup-functor of $\underline{I}_{\pi}\wedge$ inducing the identity on X	§2
$\ell(\pi^{\wedge})$	its Lie algebra	§2
$\pi:\tilde{X} \to \underline{H}$	an algebraization of the formal versal family	§3
$X(\underline{t}) = \pi^{-1}(\underline{t})$	for $\underline{t} \in \underline{H}$	§3
$\underline{H}_0$	the prorepresenting substratum of $\underline{H}$	§3
$\underline{X}_{\mu} \to \underline{H}_{\mu}$	the versal μ-constant family	§3
g_{π}: $Der_k(H) \to A^1(H,\tilde{X},O_{\tilde{X}})$	the Kodaira-Spencer map of the family $\pi : \tilde{X} \to \underline{H}$	§3
$V=V(\pi), V_{\mu}$	the kernel of the Kodaira-Spencer map	§3,4
$\{\underline{S}_{\tau}\}$	the flattening stratification of the H-module $A^1(H,\tilde{X},O_{\tilde{X}})$	§3
$\{\underline{M}_{\tau}\}$	the local moduli suite	§3
$\mu m(f)$	the μ-modality of f	§4
$K(\underline{t})$	the matrix presenting $V_{\mu} \subseteq Der_k(H_{\mu})$	§5
$L(\underline{t})$	the linear part of $K(\underline{t})$	§5

CONTENTS

INTRODUCTION

The purpose of this monograph is to contribute towards a better understanding of the local moduli problem in algebraic geometry. Let k be a field, and let X be an algebraic object, say a projective k-scheme.

The local moduli problem may then be phrased as follows. Describe the set of isomorphism classes of objects X' occuring as "arbitrary small deformations" of X.

In practice this means to define a natural filtration on the set of these isomorphism classes, such that each subset of the filtration may be given an algebraic structure, say as a k-scheme or, more generally, as an algebraic space. We shall refer to any such, natural, filtration $\{M_\tau\}$ as a local moduli suite of X.

This done, one would like to find the local structure of these new objects, their dimensions etc.

Our approach starts with a study of the formal moduli $\underline{H}^\wedge = \mathrm{Spf}(H^\wedge)$ of X, see §1 and §2.

Recall that the tangent space of $\underline{H}^\wedge$ is isomorphic to $A^1(k,X,O_X)$, the first generalized André-Quillen cohomology of X, see e.g. [La 1]. (We are painfully aware of the fact that these cohomology groups usually are denoted by $T(X)$, and are refered to as the cohomology of the cotangent complex. There are, however, good reasons not to adhere to this practice. Just look into the deformation theory for modules or Lie algebras.)

Put $\tau(X) = \dim A^1(k,X;O_X)$.

In §1 we prove that there is a unique maximal closed sub-proscheme $\underline{H}_0^\wedge = \mathrm{Spf}(H_0^\wedge)$ of $\underline{H}^\wedge$ for which the obvious composition of natural transformations

$$\mathrm{Mor}(H_0^\wedge,-) \to \mathrm{Mor}(H^\wedge,-) \to \mathrm{Def}_X$$

is injective. $\underline{H}_0^\wedge$ is the prorepresenting substratum of $\underline{H}^\wedge$.

In §2 we study the group functor $\underline{I}(\pi^\wedge)$ of those automorphisms of $H^\wedge$ that leave the formal versal family invariant, and we prove that $\underline{H}_0^\wedge$ is the fixed proscheme of the action of its Lie algebra $\ell(\pi^\wedge)$, on $\underline{H}^\wedge$.

Consider now the following conditions, see the introduction to §3,

(A_1) There exists an algebraization

$$\pi: \tilde{X} \to \underline{H} = \mathrm{Spec}(H)$$

of the formal versal family

(V) π is formally versal (see definition (3.6)).

(A_2) π is versal in the étale topology.

In §3 we prove that, under the condition (A_1) and a set of conditions, (V'), implying in particular (V) and the smoothness of $\underline{H}$, there exists a prorepresenting substratum $\underline{H}_0$ of $\underline{H}$, the formalization of which at each closed point $\underline{t} \in \underline{H}_0$ is the corresponding prorepresenting substratum of the formal moduli of $X(\underline{t}) = \pi^{-1}(\underline{t})$.

In fact, let V denote the kernel of the Kodaira-Spencer map

$$g: \mathrm{Der}_k(H) \to A^1(H,\tilde{X};O_{\tilde{X}})$$

then V is a k-sub Lie algebra of Der(H), such that the formalization coincides with $\ell(\pi^{\wedge})$. $\underline{H}_0$ is defined by the vanishing of V. Using this we prove, by glueing together the prorepresenting substrata of the versal bases corresponding to the various fibers of π, the Theorem (3.18) which asserts the existence of a local moduli suite $\{\underline{M}_\tau\}$, in the category of algebraic spaces, provided (A_1), (V') and (A_2) hold.

Put for every $\underline{t} \in \underline{H}$, $\tau(\underline{t}) = \tau(X(\underline{t}))$. Then the main result of §3 may be phrased as follows, see Theorem (3.24): Let $\underline{S}_\tau$ be the τ-constant substratum of $\underline{H}$. Then there exists a scheme theoretic quotient of an open dense subscheme $\underline{U}_\tau$ of the normalization of $\underline{S}_\tau$ by the action of the Lie algebra V, and a quasifinite dominant morphism $\underline{U}_\tau/V \to \underline{M}_\tau$. The rest of this monograph is concerned with hypersurfaces and hypersurface singularities, see §§4-7.

To illustrate the main ideas of these paragraphs, let us consider a plane affine curve defined by $f \in k[x_1,x_2]$, with only isolated singularities.

Put $X = \mathrm{Spec}(k[x_1,x_2]/(f))$. The cohomology $A^i = A^i(k,X;O_X)$ is, in this case $A^0 = \mathrm{Der}_k(k[x_1,x_2]/(f))$, $A^1 = k[x_1,x_2]/(f,\frac{\partial f}{\partial x_1},\frac{\partial f}{\partial x_2})$, $A^i = 0$ for $i \geq 2$. The formal moduli is therefore $H^{\wedge} = k[[t_{\underline{\alpha}}]]_{\underline{\alpha}\in I}$ where $I \subseteq \underline{Z}^2_+$ is such that $\{x_1^{\alpha_1}x_2^{\alpha_2}\}_{(\alpha_1,\alpha_2)\in I}$ is a basis for A^1.

There exists an algebraization

$$\pi: \tilde{X} \to \underline{H}$$

of the formal versal family of X, with $\underline{H} = \text{Spec}(H)$, $H = k[t_{\underline{\alpha}}]_{\underline{\alpha}\in I}$, $\tilde{X} = \text{Spec}(H[x_1,x_2]/(F))$, $F = f + \sum_{\underline{\alpha}\in I} t_{\underline{\alpha}} x_1^{\alpha_1} x_2^{\alpha_2}$.

Thus (A_1) is satisfied. Since $\tilde{X}$ is also a hypersurface F, we find:

(2) $A^1(H,\tilde{X};O_{\tilde{X}}) = H[x_1,x_2]/(F,\frac{\partial F}{\partial x_1},\frac{\partial F}{\partial x_2})$ is an H-module of finite type

(3) $A^i(H,\tilde{X};O_{\tilde{X}}) = 0$ for $i \geq 2$

Together with the obvious,

(1) H is k-smooth

these properties show that the conditions (V') refered to above, are satisfied.

However, the condition (A_2) is not in general satisfied. In fact, as we know, the family $x^3+y^2+t_1x+t_0$, $27t_0^2+4t_1^3 \neq 0$, is not constant even though every fiber is smooth and therefore rigid.

Consequently we may not use the technique of §3 to produce a local moduli suite for X. The conclusions of (3.18) and (3.24) are, in fact, false in the affine case.

If, however, instead of the affine scheme $\text{Spec}(k[x_1,x_2]/(f))$, we consider the formal scheme

$$X = \text{Spf}(k[[x_1,x_2]]/(f))$$

we are in a much better situation.

The correct cohomology in this case is easily computed. We find

$$A^0 = \text{Der}_k^c(k[[x_1,x_2]]/(f))$$

$$A^1 = (x_1,x_2)/(x_1,x_2)\cdot(\frac{\partial f}{\partial x_1},\frac{\partial f}{\partial x_2})+(f)$$

$$A^i = 0\ ,\quad i \geq 2$$

As in the affine case, there is an algebraization of the formal moduli, with H and F defined by the same formulas as above. The only difference is that now all the conditions (A_1), (V') and (A_2) are satisfied. Therefore we may apply the method of §3 to produce a local moduli suite for X.

Consider the Kodaira-Spencer morphism

$$g: \text{Der}_k(H) \to A^1(H,\tilde{X};O_{\tilde{X}})$$

$g(\frac{\partial}{\partial \underline{t}_\alpha})$ is the class of $\frac{\partial F}{\partial \underline{t}_\alpha}$ in

$A^1(H,\tilde{X};O_{\tilde{X}}) = (x_1,x_2)\cdot H[[x_1,x_2]]/(x_1,x_2)(\frac{\partial F}{\partial x_1},\frac{\partial F}{\partial x_2})+(F)$.

The kernel V of g is easily seen to be a k-Lie subalgebra of $\mathrm{Der}_k(H)$.

Let $\underline{H}_o := \underline{H}_o(f)$ be the closed subscheme of $\underline{H}$ on which V vanishes. $\underline{H}_o$ is the <u>prorepresenting substratum</u> of $\underline{H}$ and forms the "inner room", $\underline{M}_{\tau(f)}$ of the local moduli suite of X.

Recall the definitions of the Tjurina number $\tau(f) = \dim_k(k[[x_1,x_2]]/(f,\frac{\partial f}{\partial x_1},\frac{\partial f}{\partial x_2}))$ and, the Milnor number $\mu(f) = \dim_k(k[[x_1,x_2]]/(\frac{\partial f}{\partial x_1},\frac{\partial f}{\partial x_2}))$, associated to the singularity f.

Consider for every τ, $0 \leqslant \tau \leqslant \tau(f)$, the subscheme of $\underline{H}$

$$\underline{S}_\tau = \{\underline{t}\in\underline{H} \mid \tau(F(\underline{t}))=\tau\}$$

<u>Theorem (3.18)</u> implies the existence of a cartesian square in the category of algebraic spaces

$$(0)\qquad \begin{array}{ccc} \tilde{X}_\tau & \longrightarrow & \mathcal{X}_\tau \\ {\scriptstyle \pi_\tau}\downarrow & \square & \downarrow{\scriptstyle \bar{\pi}_\tau} \\ \underline{S}_\tau & \xrightarrow{\ \sigma_\tau\ } & \underline{M}_\tau \end{array}$$

where π_τ is the restriction of π to $\underline{S}_\tau$, and $\underline{M}_\tau$ is a glueing together of the prorepresenting substrata $\underline{H}_0(F(\underline{t}))$ for $\underline{t} \in \underline{S}_\tau$. The family $\{\underline{M}_\tau\}_{0\leqslant\tau\leqslant\tau(f)}$ is, by definition, the <u>local moduli suite of X (or f)</u>.

This notion is justified by the following facts

(1) $\bar{\pi}_\tau$ is flat

(2) If the restriction of $\bar{\pi}_\tau$ to a connected subscheme $\underline{K} \to \underline{M}_\tau$ is constant, then $\underline{K}$ is a closed point.

<u>Theorem (3.24)</u> implies that there exists an open dense subscheme $\underline{U}_\tau$ of the normalization of $\underline{S}_\tau$, and a geometric quotient $\underline{N}_\tau = \underline{U}_\tau/V_\tau$ together with a dominant quasifinite morphism

$\underline{N}_\tau \to \underline{M}_\tau$.

Here we use the fact that $\underline{S}_\tau$ is stable under V, and we have put $V_\tau = V|S_\tau$.

The example $f = x_1^5+x_2^{11}$, treated extensively in §5 and §7, shows that we may not in general assume $\underline{U}_\tau = \underline{S}_\tau$. In fact, in this example $\underline{S}_{35}/V_{35}$ does not exist as a geometric quotient.

In §§4 to 7 we restrict to the quasihomogeneous case, and we change our setting slightly by fixing the topological type of the deformations.

Let $\underline{H}_\mu = \{\underline{t}\in\underline{H}|\mu(F(\underline{t}))=\mu(f)\}$, and put $\underline{S}_{\mu\tau} = \underline{S}_\tau \cap \underline{H}_\mu$. Notice that $\underline{S}_{\mu,\tau_{min}}$ is open and dense in $\underline{H}_\mu$, where $\tau_{min} = \min\{\tau(F(\underline{t}))|\underline{t}\in\underline{H}_\mu\}$. Put also $\underline{M}_{\mu,\tau} = \sigma_\tau(\underline{S}_{\mu\tau})$. Then $\{\underline{M}_{\mu,\tau}\}_{\tau_{min}\leq\tau\leq\tau(f)}$ is the <u>local μ-constant moduli suite of X</u>.

In §4 we compute the restriction of V to $\underline{H}_\mu$. Using this we are able in §5 to compute the dim $\underline{M}_{\mu,\tau_{min}}$, and to prove

<u>Theorem (5.1)</u>: If $f = x_1^{a_1} + x_2^{a_2}$ then:

$$\dim \underline{M}_{\mu,\tau_{min}} \geq \ldots \geq \dim \underline{M}_{\mu,\tau} \geq \dim \underline{M}_{\mu,\tau+1} \geq \ldots \geq \dim \underline{M}_{\mu,\tau(f)}$$

This result shows in particular, that the generic μ-constant deformation of f represents (a component of) the generic isomorphism class of deformations of f. The example $f = x_1^3+x_2^{10}+x_3^{19}$, treated in §5, shows that the above result is not true in higher dimensions!

In §6 we turn to the study of $\underline{M}_{\mu,\tau_{min}}$. The main result is:

<u>Theorem (6.1)</u>. Let $f = x_1^{a_1}+x_2^{a_2}$ with $(a_1,a_2) = 1$, then there exists a geometric quotient $\underline{T}_{\tau_{min}} = \underline{S}_{\mu,\tau_{min}}/V$ which is a coarse moduli scheme for plane curve singularities with semigroup $\Gamma = \langle a_1,a_2\rangle$ and minimal Tjurina number.

This result has been generalized to any quasihomogeneous plane curve singularity, by B. Martin and these authors, see [L-M-P].

Finally, in §7 we treat the example $f = x^5 + y^{11}$ in detail, and in §8 we add an algorithm for computing the Kodaira-Spencer kernel V for plane curves.

The main results are also summed up in the introductions to each paragraph.

This monograph is the outgrowth of a collaboration between the two authors during the last 5 years. A first, very sketchy version appeared in 1983 [La-Pf].

Many authors have previously treated the same subject, see e.g. Merle [Me], Teissier [Z], Washburn [Wash], and Zariski [Z]. In particular Zariski in his lecture notes [Z], published in 1973, laid the foundations to the study of hypersurface singularities in the algebroid sense. Obviously, his results have influenced upon our work, even though our methods are quite different, and our goals seemingly somewhat wider.

It should be explicitly mentioned that Palamodov in [Pal] studied the notion of prorepresenting substratum (§1), and that Saito in [S], has the same calculation as we obtain in §4, of the kernel of the Kodaira-Spencer map. These ideas are, however, part of the folklore of the last decades, nourished by the work of Grothendieck, Mumford, Artin and Schlessinger, and were at the origin of one of the authors interest in this subject, see [La1].

We have profited a lot on discussions with our colleages at Berlin and Oslo.

In particular we would like to thank Dr. Bernd Martin for his contribution to §§6-8.

We are deeply indebted to Randi Møller and Inger Jansen for their skill and really infinite patience in typing the manuscript.

Finally, we are grateful for the financial support by the Humboldt-Universität zu Berlin, DDR, by the University of Oslo, Norway, and by the Norwegian Research Council, NAVF.

Given an algebraically closed field k of characteristic zero, we shall, throughout §1-§3 be concerned with an algebraic object X such as

<u>Example 1</u>. $X = \underline{c}$, a small category of k-schemes. Put

$$A^i = A^i(k, \underline{c}, O_{\underline{c}}), \quad i \geq 0, \text{ see [La 1]}.$$

Example 2. $X = \mathrm{Spec}(A)$, A any k-algebra with isolated singularities. In particular, we shall be interested in the case where $A = k[X_1,\ldots,X_n]/(f)$ is a hypersurface. In this case $A^0 = \mathrm{Der}_k(k[X_1,\ldots,X_n]/(f))$, $A^1 = k[X_1,\ldots,X_n]/(\frac{\partial f}{\partial X_1},\ldots,\frac{\partial f}{\partial X_n}, f)$ and $A^2 = 0$. For the notion of hypersurface singularity, see §4.

Example 3. $X = \underline{e}$, a small category of O_Y-Modules where Y is some k-scheme. Here $A^i = \mathrm{Ext}^i_{O_Y}(O_{\underline{e}}, O_{\underline{e}})$, $i \geq 0$ are defined as in [La 1] with Hom replacing Der. See the concluding remark, loc.cit. p. 150, of [La 1] and [La 2].

Example 4. $X = E$, a coherent $O_{\underline{P}^n}$-Module. $A^i = \mathrm{Ext}^i_{O_{\underline{P}^n}}(E,E)$, $i \geq 0$. Of particular interest is the case where E is a vector bundle on P^n.

Assume now that $\dim_k A^i < \infty$ for $i = 1,2$. Then, see e.g. [La 1], (4.2.4), there exist in all these cases a formal moduli $H^\wedge$ (a prorepresenting hull for the deformation functor) of X, and a formal versal family

$$\pi^\wedge : X^\wedge \to \mathrm{Spf}(H^\wedge) = \underline{H}^\wedge .$$

The first part of this monograph , §§1-2 is devoted to the study of $\pi^\wedge$ in this generality.

§1. THE PROREPRESENTING SUBSTRATUM OF THE FORMAL MODULI

Introduction. Let X be an object of the type considered in the main introduction above.

The basic notion in the study of local moduli of X is the notion of prorepresenting substratum of the formal moduli.

If $H^\wedge$ is the formal moduli of X, then the prorepresenting substratum $H_0^\wedge$ is the unique maximal quotient of $H^\wedge$ for which the obvious composition

$$\mathrm{Mor}(H_0^\wedge,-) \to \mathrm{Mor}(H^\wedge,-) \to \mathrm{Def}_X$$

is injective.

This quotient exists in all generality, and the object of this § is its construction, see (1.3).

In §3, we shall want to extend this notion to the algebraization of the formal moduli. It turns out that this is facilitated by the introduction of the concept of the n-th equicohomological substratum $H_{(n)}^\wedge$ of $H^\wedge$, see (1.5), and by proving, (1.6), that $H_0^\wedge$ coincides with the 0-th equicohomological substratum $H_{(0)}^\wedge$.

Let X be any algebraic object of the type discussed in the Introduction, and consider the deformation functor

$$\mathrm{Def}_X\colon \underline{\ell} \to \underline{\mathrm{Sets}},$$

the corresponding cohomology $A^i = A^i(k,X;O_X)$, $i \geq 0$ and the universal obstruction morphism

$$o_X\colon T^2 \to T^1$$

where $T^i = \mathrm{Sym}_k(A^{i*})^\wedge$. Denote by

$$H^\wedge = T^1 \underset{T^2}{\otimes} k$$

the formal moduli of X, i.e. the (prorepresenting) hull of the deformation functor Def_X, and put

$\underline{H}^{\wedge} = Spf(H^{\wedge})$.

In general there are lots of infinitesimal automorphisms of X, and non trivial obstructions for lifting these (see [Sch]). Therefore $H^{\wedge}$ does not necessarily prorepresent Def_X. However, as we shall see, there is a universal prorepresenting substratum $\underline{H}_0^{\wedge}$ of $\underline{H}^{\wedge}$, corresponding to a quotient

$$H_0^{\wedge} = H^{\wedge}/\mathfrak{a}$$

of $H^{\wedge}$.

In fact, let us consider the category $\underline{\ell}_H$ of all artinian local $H^{\wedge}$-algebras with residue field k.

Let $X^{\wedge}$ be the formal versal family on $H^{\wedge}$ defined by the identity element $1_{H^{\wedge}} \in Mor(H^{\wedge}, H^{\wedge})$ and consider the functor

$$\underline{aut}_{X^{\wedge}}: \underline{\ell}_H \to \underline{gr}$$

defined by:

$$\underline{aut}_{X^{\wedge}}(S) = \{\psi \in Aut_S(X^{\wedge} \otimes_{H^{\wedge}} S) \mid \psi \otimes_S k = 1_X\} =: aut_S(X^{\wedge} \otimes_{H^{\wedge}} S)$$

<u>Theorem (1.1)</u>. Assume $\dim_k A^i$ is countable $i = 0,1$. Then there exists a morphism of complete local $H^{\wedge}$-algebras

$$o_{\underline{a}}: H \hat{\otimes} T^1 \to H \hat{\otimes} T^0$$

such that

(i) $\quad o_{\underline{a}}(\underline{m}_{H\hat{\otimes}T^1}) \subseteq \underline{m}^2_{H\hat{\otimes}T^0}$

(ii) $\quad a_{X^{\wedge}} := (H\hat{\otimes}T^0) \otimes_{H\hat{\otimes}T^1} H^{\wedge}$

is a prorepresenting hull for the functor $\underline{aut}_{X^{\wedge}}$.

<u>Proof</u>. This follows from the proof of [La 1], (4.2.4) with $\underline{aut}_{X^{\wedge}}$ replacing Def_X and A^{i-1} replacing A^i, $i = 1,2$. Q.E.D.

Recall that there is the usual automorphism functor of X,

$$\underline{\mathrm{Aut}}_X : \underline{\mathrm{sch}}/k \to \underline{\mathrm{gr}}$$

defined by:

$$\underline{\mathrm{Aut}}_X(\underline{S}) = \mathrm{Aut}_{\underline{S}}(X \times \underline{S})$$

Assume $\underline{\mathrm{Aut}}_X$ is represented by the k-scheme $\mathrm{Aut}(X)$ and let $1 \in \mathrm{Aut}(X)$ be the identity element. Then the completion $\hat{O}_{\mathrm{Aut}(X),1}$ of the local ring of $\mathrm{Aut}(X)$ at 1, represents the fiber-functor of $\underline{\mathrm{Aut}}_X$ at $1 \in \mathrm{Aut}_k(X)$, i.e. the functor

$$\underline{\mathrm{aut}}_X : \underline{\ell} \to \underline{\mathrm{gr}}$$

defined by

$$\underline{\mathrm{aut}}_X(S) = \{\psi \in \mathrm{Aut}_S(X \otimes S) \mid \psi \otimes_S k = 1\} =: \mathrm{aut}_S(X \otimes S)$$

Let a_X be the prorepresenting hull of $\underline{\mathrm{aut}}_X$, such that with the assumption above

$$a_X \simeq \hat{O}_{\mathrm{Aut}(X),1}.$$

Notice that if $\mathrm{Aut}(X)$ is smooth, then $a_X \simeq \mathrm{Sym}_k(A^{0*})^\wedge$ (see [La1] Ch. 4).

<u>Definition (1.2)</u>. Let the ideal $\mathfrak{a} \subseteq H^\wedge$ be generated by the coefficiants of the elements of $o_{\underline{a}}(\underline{m}) \subset H \hat{\otimes} T^0$, $\underline{m}$ being the maximal ideal of $H \hat{\otimes} T^1$. Then the <u>prorepresenting substratum</u>

$$\underline{H}_0^\wedge \subseteq \underline{H}^\wedge$$

is the formal subscheme defined by $\mathfrak{a}$.

Put $H_0^\wedge = H^\wedge/\mathfrak{a}$. Then $\underline{H}_0^\wedge = \mathrm{Spf}(H_0^\wedge)$ and we shall, mildly abusing the notations, also speak about the prorepresenting substratum $H_0^\wedge$.

By construction of $o_{\underline{a}}$ it is clear that $H_0^\wedge$ is the maximal quotient of $H^\wedge$ for which

$a_{X^\wedge} \otimes_{H^\wedge} H_0^\wedge$

is $H_0^\wedge$-smooth.

Proposition (1.3). $H_0^\wedge$ is the maximal quotient of $H^\wedge$ for which the canonical morphism of functors on $\underline{\ell}$,

$\rho_0: \mathrm{Mor}(H_0^\wedge,-) \to \mathrm{Def}_X$

is injective.

Proof. Let $H_1^\wedge$ be a quotient of $H^\wedge$, and assume $\psi_1,\psi_2 \in \mathrm{Mor}(H_1^\wedge,R)$ are mapped onto the same element $\bar{\psi}_1 = \bar{\psi}_2$ in $\mathrm{Def}_X(R)$. This, of course, means that there exists an R-isomorphism $X^\wedge \otimes_{\psi_1} R \overset{\Phi}{\simeq} X^\wedge \otimes_{\psi_2} R$ where at the left side R is considered as $H^\wedge$-module via ψ_1 and at the right hand side R is considered as $H^\wedge$-module via ψ_2 .

We may assume, by induction, $\psi_1 \equiv \psi_2 \pmod{\underline{n}}$ where $\underline{n}$ is some ideal of R killed by the maximal ideal $\underline{m}_R$. Then $\Phi \otimes R/\underline{n}$ is an automorphism of $X^\wedge \otimes_{H^\wedge} R/\underline{n}$, corresponding to a morphism $a_{X^\wedge} \otimes_{H^\wedge} H_1^\wedge \to R/\underline{n}$. If $a_{X^\wedge} \otimes_{H^\wedge} H_1^\wedge$ is formally $H_1^\wedge$-smooth, then obviously this morphism may be lifted to a morphism $a_{X^\wedge} \otimes_{H^\wedge} H_1^\wedge \to R$, proving that $\Phi \otimes_R R/\underline{n}$ is liftable as an automorphism to some $\Phi_1: X^\wedge \otimes_{\psi_2} R \tilde{\to} X^\wedge \otimes_{\psi_2} R$. But then

$\Phi \circ \Phi_1^{-1}: X^\wedge \otimes_{\psi_1} R \tilde{\to} X^\wedge \otimes_{\psi_2} R$ is an isomorphism extending the identity of

$X^\wedge \otimes_{H^\wedge} R/\underline{n}$. Thus $\psi_1 = \psi_2$. From this follows that $\rho_0: \mathrm{Mor}(H_0^\wedge,-) \to \mathrm{Def}_X$ is injective.

Conversely assume $H_1^\wedge$ is a quotient of $H^\wedge$ such that $\rho_1: \mathrm{Mor}(H_1^\wedge,-) \to \mathrm{Def}_X$ is injective. If R, an object of $\underline{\ell}_H$, is an $H_1^\wedge$-algebra, then any automorphism $\Phi_{\underline{n}}$ of $X^\wedge \otimes_{H^\wedge} R/\underline{n} = (X^\wedge \otimes_{H^\wedge} H_1^\wedge) \otimes_{H_1^\wedge} R/\underline{n}$ may always be lifted to an automorphism of $X^\wedge \otimes_{H^\wedge} R$. It follows that $a_{X^\wedge} \otimes_{H^\wedge} H_1^\wedge$ has to be formally smooth, which proves the proposition.

Q.E.D.

Remark (1.4). Recall that $H^\wedge/\underline{m}^2$ represents the restriction of the deformation functor Def_X to the subcategory $\underline{\ell}_2 = \{R \in \underline{\ell} \mid \underline{m}_R^2 = 0\}$ of