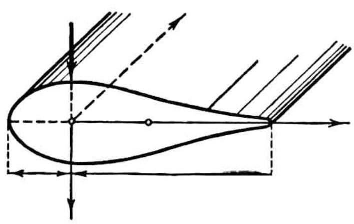
Holden-Day Series in Mathematical Physics

S. G. LEKHNITSKII

OF AN ANISOTROPIC ELASTIC BODY



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Julius J. Brandstatter, Editor

THEORY OF ELASTICITY

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Editor's Preface

This monograph, by one of the leading pioneers in the theory of elasticity of an anisotropic body,. represents some of the significant results of the author's investigations in this important field. A large class of problems in the plane theory of elasticity including those problems which are reducible to this theory are treated by the method of complex variables and other ingenious techniques.

One of the basic contributions of the author is his extension of N. I. Muskhelishvili's work in the plane theory of isotropic elasticity to the anisotropic case. Here, the author utilizes the theory of analytic functions of several complex variables in an elementary and systematic manner in order to solve some boundary value problems in an elegant and practical manner. It is of interest to point out that the theory of functions of several complex variables which ordinarily is considered to be in the domain of pure mathematics now finds application to problems in modern physics (*e.g.* dispersion relations of quantum field theory) as well as to problems of technology (*e.g.* stress and deformation of thin plate structures).

In view of the growing importance of the field of material science it is hoped that this monograph will significantly aid those scientists and engineers who are concerned with problems involving anisotropic elasticity.

In translating the monograph no liberties have been taken in altering the author's style or intent. The only deviations from the original have been the shortening of certain sentences and paragraphs which otherwise would read awkwardly if translated literally. However, this was done without sacrificing the meaning. Apart from these changes we believe that a faithful representation of the original contents has been produced.

Julius J. Brandstatter

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Author's Preface

Anisotropic materials play an important role in modern technology. Missile and aircraft designers, specialists in mining problems, solid state physicists, geophysicists, manufacturers of certain parts and materials, and in general, people engaged in the material sciences—all must deal with a variety of anisotropic problems. It is possible to produce structures which exhibit an artificial anisotropy (for example, corrugated plates and membranes made from a basic material which is elastically isotropic). Certain structures which are reinforced or strengthened by ribbing also become anisotropic. Thus, we see the importance of anisotropy in many phases of modern technology.

In the past, materials, regardless of their composition, were usually considered to be homogeneous and isotropic because such assumptions resulted in simplified calculations. Today, however, these simplified assumptions often lead to inadequate or incorrect results; our sophisticated technology requires that we take into account the anisotropy of materials, that is, the differences in elastic properties of materials in various directions.

In order to calculate the stability of anisotropic bodies which undergo elastic deformation, it is necessary to determine the stresses and deformations by theoretical means; that is, it is necessary to solve problems of the theory of elasticity of an anisotropic body. As is well known, the number of independent elastic constants in an isotropic body is equal to two (the fundamental constants are usually taken to be the Young's modulus and Poisson's ratio). In the case of an anisotropic body the number of independent elastic constants can be considerably larger, and a maximum of independent elastic constants can exist. In order to solve problems which are concerned with stress distribution and deformation in an anisotropic body, it is necessary to start with the basic equations of the theory of elasticity and to take into account the fact that the body is characterized by more than two elastic constants.

The theory of elasticity of an isotropic body has been thoroughly investigated. This is not the case, however, for the theory of elasticity of anisotropic bodies; in this latter field a great deal of literature exists in the form of monographs and articles published in different journals. Certain special problems on the state of stress and strain and the stability of anisotropic plates have received intensive study (for example, see the book *Anisotropic Plates* [18]). However, many problems of anisotropic bodies still have not been systematically examined.

We believe that it is essential to bring together this scattered material and to present it in a systematic and orderly manner. This would enable the specialists who encounter questions in the theory of elasticity of anisotropic bodies to have at their fingertips the basic material of this topic and to utilize it in their investigations.

This book contains the research and investigations of the author and some results obtained by other scientists.

The subject matter is laid out in the following way: Chapter 1 deals with the general equations of the theory of elasticity of an anisotropic body; Chapter 2 investigates the simplest cases of elastic equilibrium; Chapters 3 and 4 examine the state of stress of an anisotropic body bounded by a cylindrical surface for which the stress does not vary along the generator; Chapter 5 investigates the state of stress of an anisotropic cantilever of constant cross section deformed by a transverse force; Chapter 6 deals with the symmetric deformation and torsion of a body of revolution.

This book does not pretend to investigate all the questions of the theory of elasticity of an anisotropic body. Rather, it gives an account of certain parts of the theory of anisotropic bodies which have been studied, but not organized systematically. The book does not contain investigations on the deflection and stability of anisotropic plates because these questions are covered in the book *Anisotropic Plates*. The problems of plane deformation and generalized plane stress are discussed briefly; the most important special cases are considered in connection with more general problems. We have not touched upon the problems of the equilibrium and stability of anisotropic shells, as it is more appropriate to discuss this subject in a work connected especially with shells. We have not considered the dynamics of an elastic body (with the exception of general equations of motion). In all cases it is assumed that the deformations are elastic and small, and the material satisfies a generalized Hooke's law. Thus, we shall not consider questions of plasticity and large elastic deformations of anisotropic bodies.

The author has tried to make the exposition brief. When excessively complex calculations are not involved, the explicit formulas for stress, which can be used for calculations, are set down.

At the end of the book we have cited the literature which, in addition to works which state special problems, includes certain fundamental courses in the theory of elasticity. Numbers in brackets in the text and the footnotes indicate the literature cited in the references.

I am deeply grateful to T. V. Skvortsov for his assistance in the calculations, in drawing the graphs, and in the technical formulation of this monograph.

S. G. Lekhnitskii

Contents

Chapter 1 General equations of the theory of elasticity of an anisotropic body

1. THE STATE OF STRESS IN A CONTINUOUS SOLID BODY

In our study of the distribution of stresses and deformations in an elastic anisotropic body, we shall adopt the universal standard which regards an elastic body as a solid continuous medium.

The state of stress at any given point of a continuous body is determined entirely by the components of stress in three mutually perpendicular planes which pass through the chosen point. The planes are usually taken perpendicular to the coordinate directions of some orthogonal coordinate system. In our study we shall use, for the most part, Cartesian and cylindrical coordinates.

Let us refer the continuous body in a state of stress under the influence of some external force to a Cartesian coordinate system x, y, z, and at a given point let us consider three planes which are normal to the axes of the coordinates. The stress which acts on each area is resolved into three components along the axes. We denote the normal components (the normal stresses) by the symbol σ with an index which indicates the direction of the normal to the area. We denote the tangential components (the tangential stresses) by τ with two indices. The components of stress acting on an area normal to the x-axis are σ_x, τ_{yx}, τ_{zx}; the components of stress acting on an area normal to the y-axis are τ_{xy}, σ_y, τ_{zy}; the components of stress acting on an area normal to the z-axis are τ_{xz}, τ_{yz}, σ_z (see Figure 1). These

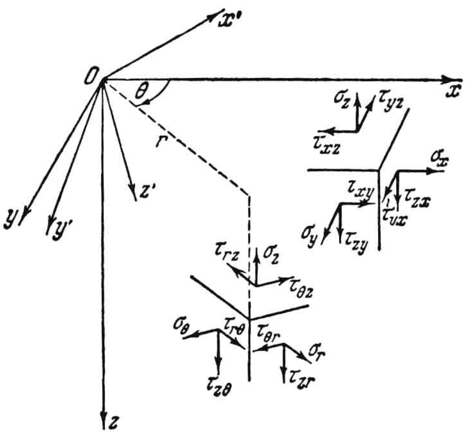

Figure 1

nine components define the stress tensor at a given point:

$$\begin{Vmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_y & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_z \end{Vmatrix};$$

the stress tensor is always symmetric because $\tau_{yx} = \tau_{xy}$; $\tau_{zy} = \tau_{yz}$, $\tau_{xz} = \tau_{zx}$ (generally, $\tau_{ij} = \tau_{ji}$ where i and j denote mutually perpendicular directions).

Let us assume that the z-axis of the Cartesian system coincides with the z-axis of a cylindrical coordinate system r, θ, z. The components of stress acting on areas normal to the coordinate directions r, θ, z are denoted respectively by: σ_r, $\tau_{\theta r}$, τ_{zr}; $\tau_{r\theta}$, σ_θ, $\tau_{z\theta}$; τ_{rz}, $\tau_{\theta z}$, σ_z (see Figure 1); in this connection $\tau_{\theta r} = \tau_{r\theta}$, $\tau_{z\theta} = \tau_{\theta z}$, $\tau_{rz} = \tau_{zr}$.

If we know the stresses in three mutually perpendicular areas, we can always determine the stress which acts on any area passing through the same point. We have the formulas:

$$\left.\begin{aligned} X_n &= \sigma_x \cos(n,x) + \tau_{xy} \cos(n,y) + \tau_{xz} \cos(n,z), \\ Y_n &= \tau_{xy} \cos(n,x) + \sigma_y \cos(n,y) + \tau_{yz} \cos(n,z), \\ Z_n &= \tau_{xz} \cos(n,x) + \tau_{yz} \cos(n,y) + \sigma_z \cos(n,z), \end{aligned}\right\} \quad [1.1]$$

where X_n, Y_n, Z_n are components of stress which act on an area with the arbitrary normal direction n.

With the aid of these formulas, we can find, by projection, the normal and tangential components of stress acting on an arbitrary area. For example, let the new Cartesian coordinate system be denoted by x', y', z'. The position of the new system

Table 1. Direction cosines

	x	y	z
x'	α_1	β_1	γ_1
y'	α_2	β_2	γ_2
z'	α_3	β_3	γ_3

with respect to the first system, x, y, z, is determined by Table 1 of direction cosines. In this table $\alpha_1 = \cos(x, x')$, $\gamma_2 = \cos(z, y')$ and so forth. By projecting $X_{x'}$, $Y_{x'}$, ..., $Z_{z'}$ in the direction of the new axes, we obtain the components of stresses in the new coordinate system (that is, in the areas normal to the x', y', and z' axes):

$$\left.\begin{aligned} \sigma'_x &= \sigma_x\alpha_1^2 + \sigma_y\beta_1^2 + \sigma_z\gamma_1^2 + 2\tau_{yz}\beta_1\gamma_1 + 2\tau_{xz}\alpha_1\gamma_1 \\ &\quad + 2\tau_{xy}\alpha_1\beta_1 , \\ \tau'_{yz} &= \sigma_x\alpha_2\alpha_3 + \sigma_y\beta_2\beta_3 + \sigma_z\gamma_2\gamma_3 + \tau_{yz}(\beta_2\gamma_3 + \beta_3\gamma_2) \\ &\quad + \tau_{xz}(\alpha_2\gamma_3 + \alpha_3\gamma_2) + \tau_{xy}(\alpha_2\beta_3 + \alpha_3\beta_2) . \end{aligned}\right\} \quad [1.2]$$

The expressions for σ'_y and σ'_z are obtained by means of a cyclic permutation of the indices α, β and γ in the first formula of [1.2]. In the second formula of [1.2] the expressions for τ'_{xz} and τ'_{xy} are obtained in the same way. The formulas for the transition from the system x', y', z' to x, y, z have the form:

$$\left.\begin{aligned} \sigma_x &= \sigma'_x\alpha_1^2 + \sigma'_y\alpha_2^2 + \sigma'_z\alpha_3^2 + 2\tau'_{yz}\alpha_2\alpha_3 + 2\tau'_{xz}\alpha_1\alpha_3 \\ &\quad + 2\tau'_{xy}\alpha_1\alpha_2 , \\ \tau_{yz} &= \sigma'_x\beta_1\gamma_1 + \sigma'_y\beta_2\gamma_2 + \sigma'_z\beta_3\gamma_3 + \tau'_{yz}(\beta_2\gamma_3 + \beta_3\gamma_2) \\ &\quad + \tau'_{xz}(\beta_1\gamma_3 + \beta_3\gamma_1) + \tau'_{xy}(\beta_1\gamma_2 + \beta_2\gamma_1) . \end{aligned}\right\} \quad [1.3]$$

We derive the expressions for the remaining components also by a cyclic permutation of the indices α, β, γ in [1.3].

Analogous formulas hold for other orthogonal coordinate systems. In the special case of formulas of type [1.2] the relations between the stresses in the Cartesian and cylindrical coordinate systems with the same z-axis are

$$\left.\begin{aligned}
\sigma_r &= \sigma_x \cos^2\theta + \sigma_y \sin^2\theta + 2\tau_{xy} \sin\theta \cos\theta \,, \\
\sigma_\theta &= \sigma_x \sin^2\theta + \sigma_y \cos^2\theta - 2\tau_{xy} \sin\theta \cos\theta \,, \\
\tau_{r\theta} &= (\sigma_y - \sigma_x) \sin\theta \cos\theta + \tau_{xy}(\cos^2\theta - \sin^2\theta) \,, \\
\tau_{rz} &= \tau_{xz} \cos\theta + \tau_{yz} \sin\theta \,, \\
\tau_{\theta z} &= -\tau_{xz} \sin\theta + \tau_{yz} \cos\theta \,, \\
\sigma_z &= \sigma_z \,.
\end{aligned}\right\} \qquad [1.4]$$

Here θ is the polar angle measured from the x-axis.

The components of stresses in a continuous body in equilibrium under the action of surface and body forces satisfy three differential equations of equilibrium. These equations expressed in Cartesian coordinates have the form:

$$\left.\begin{aligned}
\frac{\partial\sigma_x}{\partial x} + \frac{\partial\tau_{xy}}{\partial y} + \frac{\partial\tau_{xz}}{\partial z} + X &= 0 \,, \\
\frac{\partial\tau_{xy}}{\partial x} + \frac{\partial\sigma_y}{\partial y} + \frac{\partial\tau_{yz}}{\partial z} + Y &= 0 \,, \\
\frac{\partial\tau_{xz}}{\partial x} + \frac{\partial\tau_{yz}}{\partial y} + \frac{\partial\sigma_z}{\partial z} + Z &= 0 \,,
\end{aligned}\right\} \qquad [1.5]$$

where X, Y, Z are the components of the body forces (per unit of volume).