Almost Sure Convergence

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Preface

Almost sure behavior of partial sums of random variables, the subject of this book, has enjoyed both a rich classical period and a recent resurgence of research activity. This is nicely illustrated by the law of the iterated logarithm for partial sums, the subject of Chapter 5: Attempts to sharpen Borel's [1909] strong law of large numbers culminated in Khintchine's [1924] law of the iterated logarithm for Bernoulli random variables. The Kolmogorov [1929] and the Hartman–Wintner [1941] extensions of Khintchine's result to large classes of independent random variables were milestones in the classical theory. Levy's (1937, see [1954]) law of the iterated logarithm for martingales, an important class of dependent random variables, was another major advance in the classical theory.

The modern period for the law of the iterated logarithm was started by Strassen ([1964], [1965], [1966]) with his discovery of almost sure invariance principles, his deep functional law of the iterated logarithm, and his converse to the Hartman–Wintner law of the iterated logarithm (this last occurring, remarkably, a quarter century after the Hartman–Wintner result). As Chapter 5 indicates, one of the characteristics of the modern period has been an emphasis on laws of the iterated logarithm for dependent random variables.

Because of the rich history and current interest in the law of the iterated logarithm and in other areas of almost sure behavior, it seems desirable to have a monograph which treats almost sure behavior in a systematic and unified manner. This book presents such a treatment of the law of the iterated logarithm and of four other major varieties of almost sure behavior: almost sure convergence of partial sums (Chapter 2), almost sure stability of partial sums (Chapter 3), almost sure stability of weighted partial sums (Chapter 4), and recurrence of partial sums (Chapter 6).

Subdivision into major topics within chapters is usually done on the basis of dependence structure. For example, the law of the iterated logarithm for partial sums of independent random variables, which is developed in Sections 5.1–5.3 of Chapter 5, is such a major topic. The treatment of each major topic starts with more elementary and usually classical theorems and proceeds with more advanced and often recently established theorems.

Since the book's viewpoint is probabilistic, certain mathematically important topics, although logically fitting under the umbrella of almost sure behavior, are omitted because they are not probabilistic in nature. Examples of such excluded topics are the almost everywhere convergence of Fourier series and the pointwise ergodic theory of operators in L_p spaces. Topics of probabilistic interest have often been excluded as well, thereby keeping the book from being too long.

The book was developed from a course on almost sure behavior given by the author. This course was given as a sequel to the basic graduate level probability course at the University of Illinois. The book contains ample material for a one semester course. It assumes familiarity with basic real analysis and basic measure theory (such as provided by Royden's "Real Analysis") and with basic measure theoretic probability (such as provided by Ash [1972, Chapters 5–8] or Chung [1968]). The book should also prove useful for independent study, either to be read systematically or to be used as a reference.

A glossary of symbols and conventions is provided for the reader's convenience. This should be especially helpful when the book is used as a reference rather than systematically read. Results are numbered by chapter and section. For instance, Example 2.8.11 is the eleventh of the examples in Section 8 of Chapter 2. Chapters are somewhat independent; hence each may be read without extensive reference to previous chapters. Exercises are included to help the reader develop a working familiarity with the subject and to provide additional information on the subject. Numerous references are given to stimulate further reading.

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Glossary of Symbols and Conventions

$(\Omega, \mathscr{F}, P)$	Underlying probability space
EX, $E(X)$	Expectation of X
$\{X_i, i \geq 1\}$	Basic sequence of random variables
$\{S_n, n \geq 1\}$	$S_n = \sum_{i=1}^n X_i$
A^c	Complement of A
$A = B$	$P[A \cap B^c] + P[A^c \cap B] = 0$, unless explicitly stated otherwise
$A \triangle B$	$(A \cap B^c) \cup (A^c \cap B)$
$A \subset B$	$P[A \cap B^c] = 0$
A implies B	$A \subset B$
$a_n \to a$	$\lvert a \rvert = \infty$ not allowed, unless explicitly stated otherwise
On A, . . .	A implies . . .
$\mathrm{cov}(X, Y)$	Covariance of X and Y
$\mathrm{var}(Y)$	Variance of Y
R_n	n-dimensional Euclidean space
R	One-dimensional Euclidean space
R_∞	Infinite-dimensional Euclidean space
$\mathscr{C}_n$	Borel sets of R_n
$\mathscr{C}_\infty$	Borel sets of R_∞
$\mathscr{B}\{Y_\alpha, \alpha \in A\}$	σ field generated by $\{Y_\alpha, \alpha \in A\}$
$\mathscr{B}_\infty$	$\mathscr{B}\{X_1, X_2, \ldots\}$
Occurrence	See parenthetical remark, p. 1
a.s.	Almost surely
$\{\mathscr{F}_n, n \geq 1\}$ adapted to $\{Y_n, n \geq 1\}$	$\mathscr{F}_n$ increasing σ fields, $\subset \mathscr{F}$, Y_n is $\mathscr{F}_n$ measurable
$\{Y_n, \mathscr{F}_n, n \geq 1\}$ adapted stochastic sequence	Same as above
$\mathscr{F}_0$	$\{\emptyset, \Omega\}$
$I(A)$	Indicator function of event A
i.o.	Infinitely often
A_n i.o.	$\bigcap_{k=1}^\infty \bigcup_{n=k}^\infty A_n$
$\mu(X)$	Median of X
X^+	$\max(X, 0)$

X^-	$-\min(X, 0)$
$[x]$	Greatest integer function of x
P_X	Probability measure induced on $(R_1, \mathscr{C}_1)$ by X
F_X	Distribution function of X
$\alpha(X)$	Constant in definition of generalized Gaussian random variable
$a_n \sim b_n$	$\lim a_n/b_n = 1$
$a_n = o(b_n)$	$\lim a_n/b_n = 0$
$a_n = O(b_n)$	$\limsup a_n/b_n < \infty$
$\sum_{i=n+1}^{n} (\cdot)$	$= 0$
L_p	Banach space of functions with pth absolute moment
$[X \in A]$	$[\omega \mid X(\omega) \in A]$
$\equiv$	Indicates definition; e.g., $h(x) \equiv x^2$
$\log_2 x$	$\log \log x$

Contents

Chapter 3 Stability and Moments

Chapter 4 Stability of Weighted Sums of Random Variables

Chapter 5 Exponential Inequalities, the Law of the Iterated Logarithm, and Related Results

Chapter 6 Recurrence of $\{S_n, n \geq 1\}$ and Related Results

CHAPTER 1

Introduction

1.1. Delineation of the Subject

The purpose of this chapter is to introduce the reader to the subject and content of this book in some detail. The basic setting throughout the book is that of a probability space $(\Omega, \mathscr{F}, P)$ with a sequence of random variables, henceforth referred to as the basic sequence and denoted by $\{X_i, i \geq 1\}$, defined on $(\Omega, \mathscr{F}, P)$. Let $S_n = \sum_{i=1}^{n} X_i$ for $n \geq 1$. $\{S_n, n \geq 1\}$ is referred to as the sequence of partial sums.

In this book we shall study those events whose occurrence is determined by the values of infinitely many S_n. (The use of the word "occurrence," although most convenient, can cause confusion. Recall that in probabilistic terminology each $\omega \in \Omega$ is an "outcome" of an "experiment." Given a particular outcome $\omega \in \Omega$, the occurrence of an event A simply means that $\omega \in A$. Thus occurrence simply expresses membership in a set.) This remark should be made precise: For each $n \geq 1$, let $\mathscr{B}_n$ be the σ field generated by $\{X_i, 1 \leq i \leq n\}$. Thus $\mathscr{B}_n$ consists of all events of the form $[(X_1, X_2, \ldots, X_n) \in B]$ as B ranges over the Borel sets of R_n, R_n denoting an n-dimensional Euclidean space. Let $\mathscr{B}_\infty$ be the σ field generated by the $\mathscr{B}_n$. Thus $\mathscr{B}_\infty$ consists of all events of the form $[(X_1, X_2, \ldots, X_n, \ldots) \in B]$ as B ranges over the Borel sets of R_∞, R_∞ denoting an infinite-dimensional Euclidean space. Of course, $\mathscr{B}_\infty$ can also be described as consisting of all events of the form $[(S_1, S_2, \ldots, S_n, \ldots) \in B]$ as B ranges over the Borel sets of R_∞. From the viewpoint of the book this last is a more natural way to view $\mathscr{B}_\infty$. Events of $\mathscr{B}_\infty$ are of two types:

Definition 1.1.1. An event which is a member of $\mathscr{B}_n$, for some finite n, is called a weak event. An event which is a member of $\mathscr{B}_\infty$ and which is not a weak event is called a strong event. ∎

The occurrence of a weak event is determined by the values of only finitely many random variables of the basic sequence, whereas the occurrence of a strong event is determined by the values of infinitely many random variables of the basic sequence (equivalently, by the values of infinitely many random variables of the sequence of partial sums). For example, for each integer n, $[S_n \geq 0]$ is a weak event because, for a given $\omega \in \Omega$, the question of whether or not $\omega \in [S_n \geq 0]$ can be decided if the values of $X_1(\omega), X_2(\omega), \ldots, X_n(\omega)$ are known. Similarly, $[S_n \geq 0$ for infinitely many $n]$ is a strong event since, for a given $\omega \in \Omega$, for no finite n does knowing the values of $X_1(\omega), X_2(\omega), \ldots, X_n(\omega)$ determine whether $\omega \in [S_n \geq 0$ for infinitely many $n]$.

Rephrasing the statement beginning the preceeding paragraph in the light of Definition 1.1.1, in this book we study strong events. Given a strong event A of interest, the most important problem is to find conceptually and computationally simple conditions which imply that $P(A) = 1$ (or 0). In addition, we often obtain important results in which a strong event A can satisfy $0 < P(A) < 1$. Then it is an important problem to find events B of interest for which $B = A$, $A \subset B$, and $B \subset A$. For example, (letting EY denote the expectation of a random variable Y) if the X_i are martingale differences with $E \sup | X_i | < \infty$, then

$$[\sup S_n < \infty] = [S_n \text{ converges}]$$

can be proven. (Here and throughout the book, convergence means convergence to a finite limit and equality of two events means that two events contain the same outcomes with the possible exception of a null event.)

We will look at the probabilities of weak events only when the results obtained are useful in studying the probabilities of strong events. For instance, that will be our only interest in the central limit theorem. In studying the probabilities of strong events two types of assumptions are usually made concerning the basic sequence. First, the random variables of the basic sequence satisfy a dependence relationship, such as being orthogonal, being independent, being Markovian, being martingale differences, etc. Second, the random variables satisfy an absolute moment condition such as

$$\sum_{i=1}^{\infty} EX_i^2 < \infty, \qquad \sum_{i=1}^{\infty} E(X_i - EX_i)^2/i^2 < \infty,$$

$$E \sup | X_i | < \infty, \qquad \sup E | S_n | < \infty,$$

etc. A third type of assumption sometimes imposed is that the random variables of the basic sequence satisfy a stationarity condition such as being weakly stationary, identically distributed, strictly stationary, etc.

These types of assumptions combine to impose bounds on probabilities of weak events. Since the weak events form a field which generates the σ field of strong events $\mathscr{B}_\infty$, it follows by the Carathéodory extension theorem of measure theory that specification of the probabilities of the weak events uniquely determines the probabilities of the strong events. Typically, weak and general assumptions such as those discussed above impose bounds on the probabilities of certain weak events, which, in turn, imply that certain strong events of interest occur with probability zero or one.

For example, Kolmogorov's strong law of large numbers for independent random variables states that the X_i being independent and

$$\sum_{i=1}^{\infty} E(X_i - EX_i)^2/i^2 < \infty$$

together imply that

$$P[(S_n - ES_n)/n \to 0 \quad \text{as} \quad n \to \infty] = 1.$$

1.2. A Brief Chapter by Chapter Outline of Topics Covered

It seems appropriate to outline briefly the chapter by chapter organization of the book, touching on certain highlights of each chapter. In order to accomplish this, knowledge of some advanced probability concepts is assumed (e.g., martingales, Markov processes, mixing sequences). Statements concerning unfamiliar concepts should be passed over with the knowledge that such concepts will be fully developed at the appropriate places in the book.

In Chapter 2, the event $A = [S_n \text{ converges}]$ is studied. Of particular importance is the almost sure convergence of S_n (that is, $P(A) = 1$). A variety of dependence structures for the basic sequence are analyzed in Chapter 2, these structures usually implying that the X_i are orthogonal. As successively more restrictive assumptions are made concerning the dependence structure of the basic sequence, stronger and more specific results concerning the almost sure convergence of S_n are established: Suppose $ES_n^2 < \infty$ for each $n \geq 1$. Orthogonality alone guarantees that

$$\sum_{i=1}^{\infty} (\log i)^2 \, EX_i^2 < \infty$$

implies S_n converges almost surely. Under the further restriction that $\{S_n, n \geq 1\}$ is a martingale,

$$\sum_{i=1}^{\infty} EX_i^2 < \infty \qquad (\text{indeed } \sup E \mid S_n \mid < \infty)$$

implies that S_n converges almost surely. Under the still further restriction that the X_i are independent with $EX_i = 0$ for each $i \geq 1$, Kolmogorov's three series theorem characterizes the almost sure convergence of S_n in terms of the convergence of three numerical series. A major part of Chapter 2 is devoted to this progressive restricting of the assumptions concerning the dependence structure. The case of independence, for which most of the major results are classical, and the martingale case, for which many of the major results are the consequence of recent research, both receive particular emphasis in Chapter 2. In the martingale case questions concerning local convergence are carefully examined. For example,

$$\left[\sum_{i=1}^{\infty} E(X_i^2 \mid X_1, X_2, \ldots, X_{i-1}) < \infty\right] = [S_n \text{ converges}]$$

is shown under the assumption that the X_i are martingale differences with $E \sup X_i^2 < \infty$. The question of the almost sure convergence of S_n being implied by absolute moment conditions on the S_n without any particular dependence structure assumed for the basic sequence is analyzed also. Several applications to real analysis are made in Chapter 2. For example, several results about the almost everywhere convergence of Haar series are shown to follow easily from martingale convergence results.

Even if a sequence of random variables $\{T_n, n \geq 1\}$ is almost surely divergent (that is, $P[T_n \text{ converges as } n \to \infty] = 0$) it is easy to show that there exists constants $a_n \to \infty$ and b_n such that T_n is stabilized; that is $(T_n - b_n)/a_n$ converges almost surely to zero. In Chapter 3, the stability of S_n is studied when S_n is almost surely divergent. The case when the X_i are independent and identically distributed is studied first: As successively higher absolute moments of X_1 are assumed finite, sequences $\{a_n, n \geq 1\}$ converging successively slower to infinity are shown to stabilize S_n. According to Kolmogorov's strong law of large numbers for independent identically distributed random variables, if $E \mid X_1 \mid < \infty$, then

$$(S_n - nEX_1)/n \to 0 \qquad \text{almost surely.}$$

If, in addition, $E \mid X_1 \mid^p < \infty$ for some $1 < p < 2$, then

$$(S_n - nEX_1)/n^{1/p} \to 0 \qquad \text{almost surely.}$$

Let var Y denote the variance of a random variable Y. If, in addition, $EX_1^2 < \infty$, then

$$\limsup \frac{(S_n - nEX_1)}{[2n \operatorname{var}(X_1) \log\log(n \operatorname{var} X_1)]^{1/2}} \leq 1 \quad \text{almost surely.}$$

Stability is also studied when the X_i are independent but not necessarily identically distributed, when the X_i are martingale differences, when the X_i are strictly stationary, when the X_i are mixing, when the S_n are Markovian, and when the S_n are restricted by absolute moment conditions without any particular dependence structure assumed for the basic sequence. One such important result is the pointwise ergodic theorem: If $\{X_i, i \geq 1\}$ is strictly stationary and ergodic with $E|X_1| < \infty$, then

$$(S_n - nEX_1)/n \to 0 \quad \text{almost surely.}$$

A second important result is given by: If $\{X_i, i \geq 1\}$ is a martingale difference sequence with

$$\sum_{i=1}^{\infty} E|X_i|^p/i^{1+p/2} < \infty \quad \text{for some} \quad p \geq 2,$$

then

$$S_n/n \to 0 \quad \text{almost surely.}$$

A third important result is given by: If $\{X_i, i \geq 1\}$ is an independent sequence with $E|X_i|^{1+\delta} \leq K < \infty$ for some $\delta > 0$ and all $i \geq 1$, then

$$(S_n - ES_n)/n \to 0 \quad \text{almost surely.}$$

As discussed in the preceding paragraphs, the almost sure convergence of $(S_n - b_n)/a_n$ to zero is studied in Chapter 3. The more general problem of the almost sure convergence of centered weighted sums $T_n = \sum_{k=1}^{\infty} a_{nk}X_k - b_n$ to zero is studied in Chapter 4. (The a_{nk} are the weights and the b_n are the centering constants.) The purpose is to prove almost sure convergence results for as broad a class of coefficient matrices $\{a_{nk}\}$ as possible. This is in contrast to Chapter 3 where $\{a_{nk}\}$ is always assumed to have a specific structure (for example, $a_{nk} = n^{-1}$ for $k \leq n$, $a_{nk} = 0$ for $k > n$). The case when the X_i are independent and identically distributed is studied most, but the cases when the X_i are independent but not necessarily identically distributed, when the X_i are martingale differences, when the X_i are strictly stationary, and when the X_i are strongly multiplicative receive attention too. The following result is typical of Chapter 4: Let

$$\sum_{k=1}^{\infty} a_{nk}^2 \le Cn^{-\delta} \quad \text{for some} \quad C < \infty, \quad \delta > 0 \quad \text{and all} \quad n \ge 1,$$
$$a_{nk}^2 \le Ck^{-1} \quad \text{for some} \quad C < \infty \quad \text{and all} \quad n \ge 1, \quad k \ge 1,$$

and the X_i be independent identically distributed with $EX_1 = 0$ and $EX_1^2 < \infty$. Then

$$\sum_{k=1}^{\infty} a_{nk} X_k \to 0 \quad \text{almost surely as} \quad n \to \infty.$$

In the special case of weighted averages ($T_n = \sum_{i=1}^n a_i X_i / \sum_{i=1}^n a_i$) with strictly positive weights, more precise results are shown. For example, $\{X_i, i \ge 1\}$ independent identically distributed with $E \mid X_1 \mid \log^+ \mid X_1 \mid < \infty$, $EX_1 = 0$, and $\{a_i, i \ge 1\}$ uniformly bounded implies $T_n \to 0$ almost surely.

In Chapter 5, the law of the iterated logarithm and closely related results are studied. The case when the X_i are independent is studied most. For this case the classical exponential inequalities approach is used to derive Kolmogorov's well-known law of the iterated logarithm. Strassen's almost sure invariance principle is used to derive the Hartman–Wintner law of the iterated logarithm. One of the most interesting results of Chapter 5 is this Hartman–Wintner law of the iterated logarithm combined with its converse due to Strassen: Let the X_i be independent identically distributed. Then

$$\limsup \frac{S_n}{(2n\, c \log \log n)^{1/2}} = 1 \quad \text{almost surely for some} \quad 0 < c < \infty$$

if and only if $EX_1 = 0$ and $0 < EX_1^2 = c < \infty$. This delicate result about the magnitude of the asymptotic fluctuations of the S_n is typical of the results of Chapter 5 in that it contains remarkably precise information and is quite difficult to prove. Besides the case of independence, the law of the iterated logarithm is also studied when the X_i are martingale differences, mixing, and strongly multiplicative. In the case of mixing sequences, a central limit theorem with an error estimate is used to derive the needed probability inequalities. This, along with Kolmogorov's and Strassen's approach referred to above, is one of the major approaches for deriving laws of the iterated logarithm. A solution to the problem of finding necessary and sufficient conditions for the strong law of large numbers in the case of independence is given also; this is a by-product of the study of the exponential inequalities mentioned above.

Given a sequence of positive constants $\{a_n, n \geq 1\}$, let $T_n = \sum_{i=1}^{n} X_i/a_n$ for each $n \geq 1$. In Chapter 6 the recurrence of $\{T_n, n \geq 1\}$ and related results are studied. The most important problem considered is the determination of the recurrent states of $\{S_n, n \geq 1\}$. That is, for which $c \in R_1$ does

$$P[S_n \in (c - \varepsilon, c + \varepsilon) \text{ for infinitely many } n] = 1$$

hold for each $\varepsilon > 0$? This question is studied when the X_i are independent identically distributed and more generally when $\{S_n, n \geq 1\}$ is Markovian with stationary transition probabilities. The question is also studied when the X_i are independent identically distributed random vectors taking values in an n-dimensional lattice. An important result of Chapter 6 is that the X_i independent identically distributed nonlattice random variables with $EX_1 = 0$ implies that each real number c is a recurrent state of $\{S_n, n \geq 1\}$. Assuming that the X_i are independent identically distributed, the determination of the recurrent states of $\{T_n, n \geq 1\}$ for $a_n = n^\alpha$, $0 < \alpha \leq 1$ fixed, is studied. For example, $EX_1 = 0$ and $a_n = n^{1/2}$ implies that $-\infty$ and ∞ are recurrent states of $\{T_n, n \geq 1\}$. That is,

$$\limsup S_n/n^{1/2} = \infty \qquad \text{and} \qquad \liminf S_n/n^{1/2} = -\infty$$

almost surely. Assuming that the X_i are independent identically distributed, certain questions concerning the amount of time that $\{S_n, n \geq 1\}$ spends in various subsets of R_1 are also studied.

1.3. Methodology

Certain remarks about the methodology used in proving results about the occurrence of strong events seems appropriate. Typically, proofs tend to be a mixture of two kinds of analyses. First, magnitudes of probabilities are estimated using mostly elementary techniques of classical real analysis. The inequalities of Chebyshev, Holder, and Jensen, integration and summation by parts, splitting integrals into pieces, Taylor series expansions, etc., are heavily drawn upon. Second, the clever use of certain probabilistic and measure-theoretic techniques translate these estimates of probabilities into statements about the probability of occurrence of strong events of interest. The Borel–Cantelli lemma, truncation of random variables, centering at means or medians, stopping rule techniques, etc., are heavily drawn upon. Combinatorial arguments are sometimes useful. Complex analysis and functional analysis, although occasionally used, do not play a major role. One of the keys to understanding the choice of methodology is that probabilities are seldom computed but rather estimated. Indeed, the generality of typical hypotheses (for example, X_i independent identically distributed with $EX_1 = 0$) prohibits the computation of probabilities of weak events.

In many proofs a major step is the establishment of a maximal inequality. For example, $\{X_i, i \geq 1\}$ orthogonal implies that

$$E[\max_{i \leq n} S_i^2] \leq (\log 4n/\log 2)^2 \sum_{i=1}^{n} EX_i^2 \quad \text{for all} \quad n \geq 1,$$

an inequality which plays a major role in the analysis of the orthogonal case.

Often certain techniques become associated with a particular dependence structure. For example, stopping rule techniques play a major role in the study of the local convergence of martingales and truncation plays a major role in problems where the X_i are independent.

It is a major purpose of this book to stress methods of proof as well as present interesting results concerning almost sure behavior.

1.4. Applications to Fields outside Probability

Besides being of intrinsic interest to probabilists, the results presented in the book have applications to number theory, real analysis, and statistics. In this section we sketch three such applications.

The most famous example of application to number theory is the result that except for a set of Lebesque measure zero, all real numbers in the unit interval are *normal* in the sense that they have decimal expansions in which the digits 0, 1, . . . , 9 occur with equal limiting relative frequency. This follows immediately from Kolmogorov's strong law of large numbers for independent identically distributed random variables.

Certain probabilistic results concerning almost sure convergence have application in real analysis to the almost everywhere convergence of certain orthogonal series of real functions. For example, since the successive partial sums of a Haar series form a martingale satisfying a certain regularity condition, certain rather deep results concerning the almost everywhere convergence of Haar series follow rather easily from the study of martingale convergence presented in Chapter 2. Results in probability theory (such as the law of the iterated logarithm) sometimes suggest results for certain classical types of orthogonal series even when the probabilistic proofs do