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The Fokker-Planck Equation

Methods of Solution and Applications

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With 95 Figures

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Foreword

One of the central problems synergetics is concerned with consists in the study of macroscopic qualitative changes of systems belonging to various disciplines such as physics, chemistry, or electrical engineering. When such transitions from one state to another take place, fluctuations, i.e., random processes, may play an important role.

Over the past decades it has turned out that the Fokker-Planck equation provides a powerful tool with which the effects of fluctuations close to transition points can be adequately treated and that the approaches based on the Fokker-Planck equation are superior to other approaches, e.g., based on Langevin equations. Quite generally, the Fokker-Planck equation plays an important role in problems which involve noise, e.g., in electrical circuits.

For these reasons I am sure that this book will find a broad audience. It provides the reader with a sound basis for the study of the Fokker-Planck equation and gives an excellent survey of the methods of its solution. The author of this book, Hannes Risken, has made substantial contributions to the development and application of such methods, e.g., to laser physics, diffusion in periodic potentials, and other problems. Therefore this book is written by an experienced practitioner, who has had in mind explicit applications to important problems in the natural sciences and electrical engineering.

This book may be seen in the context of the book by C. W. Gardiner, "Handbook of Stochastic Methods", in this series which gives a broad and detailed overview of stochastic processes, and of the book by W. Horsthemke and R. Lefever, "Noise-induced Transitions", which treats a problem of particular current interest, namely, multiplicative noise.

Readers who are interested in learning more about the connection between the Fokker-Planck equation and other approaches within the frame of synergetics are referred to my introductory text "Synergetics. An Introduction".

H. Haken

Preface

Fluctuations are a very common feature in a large number of fields. Nearly every system is subjected to complicated external or internal influences that are not fully known and that are often termed noise or fluctuations. The Fokker-Planck equation deals with those fluctuations of systems which stem from many tiny disturbances, each of which changes the variables of the system in an unpredictable but small way. The Fokker-Planck equation was first applied to the Brownian motion problem. Here the system is a small but macroscopic particle, immersed in fluid. The molecules of the fluid kick around the particle in an unpredictable way so the position of the particle fluctuates. Because of these fluctuations we do not know its position exactly, but instead we have a certain probability to find the particle in a given region. With the Fokker-Planck equation such a probability density can be determined. This equation is now used in a number of different fields in natural science, for instance in solid-state physics, quantum optics, chemical physics, theoretical biology and circuit theory.

This book deals with the derivation of the Fokker-Planck equation, the methods of solving it, and some of its applications. Whereas for some cases (e.g., linear problems, stationary problems with only one variable) the Fokker-Planck equation can be solved analytically, it is in general very difficult to obtain a solution. Various methods for solving the Fokker-Planck equation such as the simulation method, eigenfunction expansion, numerical integration, the variational method and the matrix continued-fraction method will be discussed. The last method especially, which turns out to be very effective in dealing with simple Fokker-Planck equations having two variables, is treated in detail. As far as I know it has not yet been presented in review or book form.

In the last part of the book the methods for solving the Fokker-Planck equation are applied to the statistics of a simple laser model and to Brownian motion in potentials, especially in periodic potentials. By investigating the statistical properties of laser light, I first became acquainted with the Fokker-Planck equation and I soon learned to appreciate it as a powerful tool for treating the photon statistics of lasers and the statistics of other nonlinear systems far from thermal equilibrium.

The main emphasis in the applications is made to the problem of Brownian motion in periodic potentials. This problem occurs, for instance, in solid-state physics (Josephson tunneling junction, superionic conductor), chemical physics (infrared absorption by rotating dipoles) and electrical circuit theory (phase-locked loops). Whereas the Fokker-Planck equation for this problem was solved

many years ago for the overdamped case (large friction), solutions for arbitrary friction have been obtained only recently. It will be shown that the solution of the corresponding Fokker-Planck equation for Brownian motion in periodic potentials (as well as for other potentials) can be expressed in terms of matrix continued fractions which are very suitable for computer evaluation.

The present book is based on seminar and lecture notes, prepared and presented at the University of Ulm. Hopefully this book will be useful for graduate students in physics, chemical physics and electrical engineering to get acquainted with the Fokker-Planck equation and the methods of solving it, and that some parts of it will also be profitable to the research worker in these fields.

I wish to thank Prof. H. Haken for inviting me to write this monograph for the *Springer Series in Synergetics*. As a co-worker of Prof. Haken for nearly ten years, I had the privilege to work with him and his group in a very stimulating and creative atmosphere. I also want to express my gratitude to Dr. H. Lotsch and his staff of the Springer-Verlag for their co-operation. Next I wish to thank my co-worker and colleague Dr. H. D. Vollmer. Most of my research on the Fokker-Planck equation was done in close collaboration with him. With only few exceptions he has also provided me with the numerical results presented in this book. Furthermore, he has made many suggestions for improving the manuscript. The help of Dipl. Phys. P. Jung, Dr. M. Mörsch, and Dipl. Phys. K. Voigtländer for preparing the figures and for reading the proofs is also greatly appreciated. Last but not least I wish to thank Mrs. I. Gruhler and Mrs. H. Wenning for skilfully and patiently typing and correcting the manuscript.

Ulm, February 1984 *H. Risken*

Contents

1. Introduction

A Fokker-Planck equation was first used by *Fokker* [1.1] and *Planck* [1.2] to describe the Brownian motion of particles. To become familiar with this equation we first discuss the Brownian motion of particles in its simplest form.

1.1 Brownian Motion

1.1.1 Deterministic Differential Equation

If a small particle of mass m is immersed in a fluid a friction force will act on the particle. The simplest expression for such a friction or damping force is given by Stokes' law

$$F_c = -\alpha v\,. \tag{1.1}$$

Therefore the equation of motion for the particle in the absence of additional forces reads

$$m\dot{v} + \alpha v = 0 \tag{1.2}$$

or

$$\dot{v} + \gamma v = 0\,; \qquad \gamma = \alpha/m = 1/\tau\,. \tag{1.3}$$

Thus an initial velocity $v(0)$ decreases to zero with the relaxation time $\tau = 1/\gamma$ according to

$$v(t) = v(0)\,\mathrm{e}^{-t/\tau} = v(0)\,\mathrm{e}^{-\gamma t}\,. \tag{1.4}$$

The physics behind the friction is that the molecules of the fluid collide with the particle. The momentum of the particle is transferred to the molecules of the fluid and the velocity of the particle therefore decreases to zero. The differential equation (1.3) is a deterministic equation, i.e., the velocity $v(t)$ at time t is completely determined by its initial value according to (1.4).

1.1.2 Stochastic Differential Equation

The deterministic equation (1.2) is valid only if the mass of the particle is large so that its velocity due to thermal fluctuations is negligible. From the equipartition law, the mean energy of the particle is (in one dimension)

$$\tfrac{1}{2}m\langle v^2\rangle = \tfrac{1}{2}kT\,, \tag{1.5}$$

where k is Boltzmann's constant and T is the temperature. For smaller mass m the thermal velocity $v_{\text{th}} = \sqrt{\langle v^2\rangle} = \sqrt{kT/m}$ may be observable and therefore the velocity of a "small" particle cannot be described exactly by (1.3) with the solution (1.4). If the mass of the small particle is still large compared to the mass of the molecules, one expects (1.2) to be valid approximately. Equation (1.2) must, however, be modified so that it leads to the correct thermal energy (1.5). The modification consists in adding a fluctuating force $F_{\text{f}}(t)$ on the right-hand side of (1.2), i.e., the total force of the molecules acting on the small particle is decomposed into a continuous damping force $F_{\text{c}}(t)$ and a fluctuating force $F_{\text{f}}(t)$ according to [1.3].

$$F(t) = F_{\text{c}}(t) + F_{\text{f}}(t) = -\alpha v(t) + F_{\text{f}}(t)\,. \tag{1.6}$$

This force $F_{\text{f}}(t)$ is a stochastic or random force, the properties of which are given only in the average.

We now want to discuss why a stochastic force occurs. If we were to treat the problem exactly, we should have to solve the coupled equations of motion for all the molecules of the fluid and for the small particle, and no stochastic force would occur. Because of the large number of molecules in the fluid (the number is of the order 10^{23}), however, we cannot generally solve these coupled equations. Furthermore, since we do not know the initial values of all the molecules of the fluid, we cannot calculate the exact motion of the small particle immersed in the fluid. If we were to use another system (particle and fluid) identical to the first except for the initial values of the fluid, a different motion of the small particle results. As usually done in thermodynamics, we consider an ensemble of such systems (Gibbs ensemble). The force $F_{\text{f}}(t)$ then varies from system to system and the only thing we can do is to consider averages of this force for the ensemble.

Inserting (1.6) into (1.2) and dividing by the mass we get the equation of motion

$$\dot{v} + \gamma v = \Gamma(t)\,. \tag{1.7}$$

Here we have introduced the fluctuating force per unit mass

$$\Gamma(t) = F_{\text{f}}(t)/m\,, \tag{1.8}$$

which is called the Langevin force. Equation (1.7) is called a stochastic differential equation because it contains the stochastic force $\Gamma(t)$.

To proceed further one has to know some properties of this Langevin force $\Gamma(t)$. First we assume that its average over the ensemble should be zero

$$\langle \Gamma(t) \rangle = 0 , \tag{1.9}$$

because the equation of motion of the average velocity $\langle v(t) \rangle$ should be given by (1.2). If we multiply two Langevin forces at different times we assume that the average value is zero for time differences $t' - t$ which are larger than the duration time τ_0 of a collision, i.e.,

$$\langle \Gamma(t)\Gamma(t') \rangle = 0 \quad \text{for} \quad |t - t'| \geqq \tau_0 . \tag{1.10}$$

This assumption seems to be reasonable, because the collisions of different molecules of the fluid with the small particle are approximately independent. Usually, the duration time τ_0 of a collision is much smaller than the relaxation time $\tau = 1/\gamma$ of the velocity of the small particle. We may therefore take the limit $\tau_0 \to 0$ as a reasonable approximation, giving

$$\langle \Gamma(t)\Gamma(t') \rangle = q\,\delta(t - t') . \tag{1.11}$$

The δ function appears because otherwise the average energy of the small particle cannot be finite as it should be according to the equipartition law (1.5). This will be discussed in detail in Sect. 3.1, where it is furthermore shown that the noise strength q of the Langevin force is then given by

$$q = 2\gamma kT/m . \tag{1.12}$$

To determine higher correlations like $\langle v(t_1)v(t_2)\ldots v(t_n) \rangle$ higher correlations of $\Gamma(t)$ must be known. One usually assumes that the $\Gamma(t)$ have a Gaussian distribution with δ correlation (Chap. 3). By integrating the Langevin equation (1.7) and by using (1.9, 11, 12) we can calculate the diffusion constant (Chap. 3). As is well known, this diffusion constant was first obtained by *Einstein* [1.4], who initiated the term theory of Brownian motion.

A noise force with the δ correlation (1.11) is called white noise, because the spectral distribution (Sect. 2.4.3) which is given by the Fourier transform of (1.11) is then independent of the frequency ω. If the stochastic forces $\Gamma(t)$ are not δ correlated, i.e., if the spectral density depends on the frequency, one uses the term colored noise.

1.1.3 Equation of Motion for the Distribution Function

Because in (1.7) $\Gamma(t)$ varies from system to system in the ensemble, i.e., it is a stochastic quantity, the velocity will also vary from system to system, i.e., it will become a stochastic quantity, too. We therefore may ask for the probability to find the velocity in the interval $(v, v + dv)$, or in other words we may ask for the

number of systems of the ensemble whose velocities are in the interval $(v, v+\mathrm{d}v)$ divided by the total number of systems in the ensemble. Because v is a continuous variable we may ask for the probability density $W(v)$, also often called probability distribution in the physical literature. The probability density times the length of the interval $\mathrm{d}v$ is then the probability of finding the particle in the interval $(v, v+\mathrm{d}v)$. This distribution function depends on time t and the initial distribution. The equation of motion for the distribution function $W(v,t)$ (Chap. 4) is given by

$$\frac{\partial W}{\partial t} = \gamma \frac{\partial (v\,W)}{\partial v} + \gamma \frac{kT}{m} \frac{\partial^2 W}{\partial v^2} . \tag{1.13}$$

Equation (1.13) is one of the simplest Fokker-Planck equations. By solving (1.13) starting with $W(v,0)$ for $t=0$ and subject to the appropriate boundary conditions, one obtains the distribution function $W(v,t)$ for all later times. Once we have found $W(v,t)$, any averaged value of the velocity can be calculated by integration [$h(v)$ arbitrary function of v]

$$\langle h(v(t)) \rangle = \int_{-\infty}^{\infty} h(v)\, W(v,t)\, \mathrm{d}v . \tag{1.14}$$

As shown in Sect. 4.7.2, averaged values for multi-time functions may also, for certain processes, by evaluated by use of appropriate solutions of (1.13).

1.2 Fokker-Planck Equation

In this introductory chapter it is mainly discussed how a Fokker-Planck equation and some special forms of it look, how they arise and where and how one may use the Fokker-Planck equation. Many review articles and books on the Fokker-Planck equation exist [1.5 – 15].

1.2.1 Fokker-Planck Equation for One Variable

In Sect. 1.1 we found an equation of motion for the distribution function $W(v,t)$ for one-dimensional Brownian motion. As mentioned, it is a special Fokker-Planck equation. The general Fokker-Planck equation for one variable x has the form

$$\frac{\partial W}{\partial t} = \left[-\frac{\partial}{\partial x} D^{(1)}(x) + \frac{\partial^2}{\partial x^2} D^{(2)}(x) \right] W . \tag{1.15}$$

In (1.15) $D^{(2)}(x) > 0$ is called the diffusion coefficient and $D^{(1)}(x)$ the drift coefficient. The drift and diffusion coefficients may also depend on time. Equation (1.13) is seen to be a special Fokker-Planck equation where the drift coefficient is linear and the diffusion coefficient is constant. Equation (1.15) is an equation of motion for the distribution function $W(x,t)$. Mathematically, it is a linear second-order partial differential equation of parabolic type. Roughly speaking, it is a diffusion equation with an additional first-order derivative with respect to x. In the mathematical literature, (1.15) is also called a forward Kolmogorov equation.

1.2.2 Fokker-Planck Equation for N Variables

A generalization of (1.15) to the N variables $x_1 \ldots x_N$ has the form

$$\frac{\partial W}{\partial t} = \left[-\sum_{i=1}^{N} \frac{\partial}{\partial x_i} D_i^{(1)}(\{x\}) + \sum_{i,j=1}^{N} \frac{\partial^2}{\partial x_i \partial x_j} D_{ij}^{(2)}(\{x\}) \right] W . \tag{1.16}$$

The drift vector $D_i^{(1)}$ and the diffusion tensor $D_{ij}^{(2)}$ generally depend on the N variables $x_1, \ldots, x_N = \{x\}$. The Fokker-Planck equation (1.16) is an equation for the distribution function $W(\{x\}, t)$ of N macroscopic variables $\{x\}$. (Here x_i may be variables of different kinds for instance position and velocity.)

1.2.3 How Does a Fokker-Planck Equation Arise?

As discussed already for the Brownian motion case, the complete solution of a macroscopic system would consist in solving all the microscopic equations of the system. Because we cannot generally do this we use instead a stochastic description, i.e., we describe the system by macroscopic variables which fluctuate in a stochastic way. The Fokker-Planck equation is just an equation of motion for the distribution function of fluctuating macroscopic variables. For a deterministic treatment we neglect the fluctuations of the macroscopic variables. For the Fokker-Planck equation (1.16) this would mean that we neglect the diffusion term.

Equation (1.16) is then equivalent to the system of differential equations ($i = 1, \ldots, N$)

$$\mathrm{d}x_i/\mathrm{d}t = \dot{x}_i = D_i^{(1)}(x_1, \ldots, x_N) = D_i^{(1)}(\{x\}) \tag{1.17}$$

for the N macrovariables $\{x\}$. Table 1.1 gives a schematic representation of the following three stages of treating a system. A rigorous derivation of stochastic treatment should start with microscopic description. The deterministic treatment should then follow from the stochastic treatment by neglecting the fluctuations, as indicated by the big arrows. The drift and diffusion coefficients $D_i^{(1)}$ and $D_{ij}^{(2)}$ especially should be derived rigorously from the microscopic equations. Such a rigorous derivation may be very complicated or even impossible. In this case, one