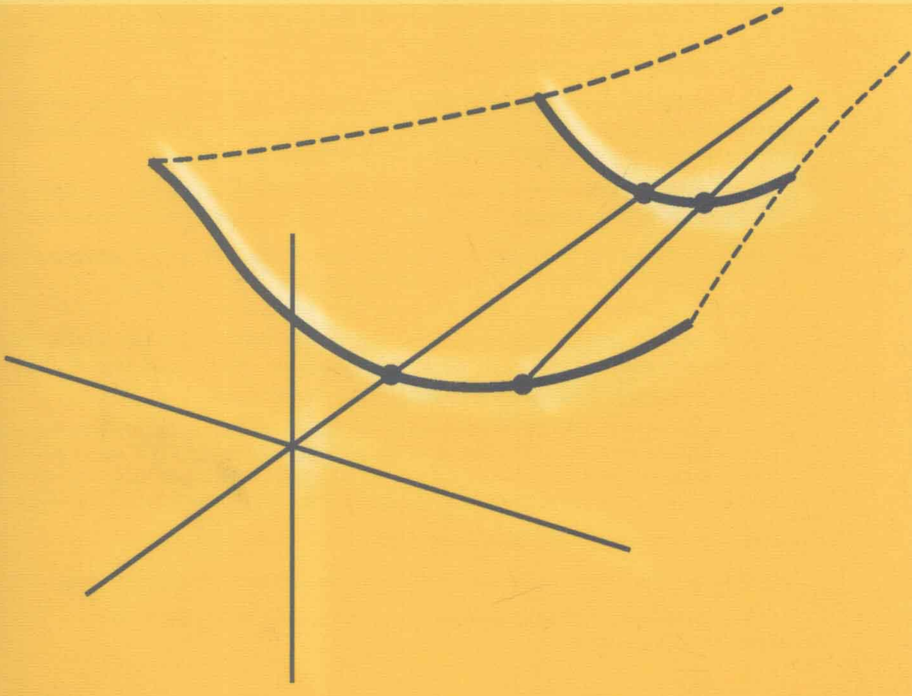
Lecture Notes in Mathematics

1926

Luis Barreira
Claudia Valls

Stability of Nonautonomous Differential Equations



Springer

Luis Barreira · Claudia Valls

Stability of Nonautonomous Differential Equations

Springer

Authors

Luis Barreira
Claudia Valls

Departamento de Matemática
Instituto Superior Técnico
Av. Rovisco Pais
1049-001 Lisboa
Portugal

e-mail: barreira@math.ist.utl.pt
cvalls@math.ist.utl.pt
URL: http://www.math.ist.utl.pt/~barreira/

Library of Congress Control Number: 2007934028

Mathematics Subject Classification (2000): 34Dxx, 37Dxx

ISSN print edition: 0075-8434
ISSN electronic edition: 1617-9692
ISBN 978-3-540-74774-1 Springer Berlin Heidelberg New York
DOI 10.1007/978-3-540-74775-8

Springer is a part of Springer Science+Business Media
springer.com

Typesetting by the authors and SPi using a Springer LaTeX macro package

Cover design: *design & production* GmbH, Heidelberg

Printed on acid-free paper SPIN: 12114993 41/SPi 5 4 3 2 1 0

To our parents

Preface

The main theme of this book is the stability of nonautonomous differential equations, with emphasis on the study of the existence and smoothness of invariant manifolds, and the Lyapunov stability of solutions. We always consider a nonuniform exponential behavior of the linear variational equations, given by the existence of a nonuniform exponential contraction or a nonuniform exponential dichotomy. Thus, the results hold for a much larger class of systems than in the "classical" theory of exponential dichotomies.

The departure point of the book is our joint work on the construction of invariant manifolds for nonuniformly hyperbolic trajectories of nonautonomous differential equations in Banach spaces. We then consider several related developments, concerning the existence and regularity of topological conjugacies, the construction of center manifolds, the study of reversible and equivariant equations, and so on. The presentation is self-contained and intends to convey the full extent of our approach as well as its unified character. The book contributes towards a rigorous mathematical foundation for the theory in the infinite-dimensional setting, also with the hope that it may lead to further developments in the field. The exposition is directed to researchers as well as graduate students interested in differential equations and dynamical systems, particularly in stability theory.

The first part of the book serves as an introduction to the other parts. After giving in Chapter 1 a detailed introduction to the main ideas and motivations behind the theory developed in the book, together with an overview of its contents, we introduce in Chapter 2 the concept of nonuniform exponential dichotomy, which is central in our approach, and we discuss some of its basic properties. Chapter 3 considers the problem of the robustness of nonuniform exponential dichotomies.

In the second part of the book we discuss several consequences of local nature for a nonlinear system when the associated linear variational equation admits a nonuniform exponential dichotomy. In particular, we establish in Chapter 4 the existence of Lipschitz stable manifolds for nonautonomous equations in a Banach space. In Chapters 5 and 6 we establish the smoothness of the stable manifolds. We first consider the finite-dimensional case in Chapter 5, with the method of invariant families of cones. This approach uses in a decisive manner the compactness of the closed unit ball in the ambient space, and this is why we consider only finite-dimensional spaces in this chapter. Moreover, the proof strongly relies on the use of Lyapunov norms to control the nonuniformity of the exponential dichotomies. As an outcome of our approach we provide examples of C^1 vector fields with invariant stable manifolds, while in the existing nonuniform hyperbolicity theory one assumes that the vector field is of class $C^{1+\alpha}$. In Chapter 6 we consider differential equations in Banach spaces, although at the expense of slightly stronger assumptions for the vector field. The method of proof is different from the one in Chapter 5, and is based on the application of a lemma of Henry to obtain both the existence and smoothness of the stable manifolds using a single fixed point problem. In addition, we show that not only the trajectories but also their derivatives with respect to the initial condition decay with exponential speed along the stable manifolds. A feature of our approach is that we deal directly with flows or semiflows instead of considering the associated time-1 maps. In Chapter 7 we establish a version of the Grobman–Hartman theorem for nonautonomous differential equations in Banach spaces, assuming that the linear variational equation admits a nonuniform exponential dichotomy. In addition, we show that the conjugacies that we construct are always Hölder continuous.

The third part of the book is dedicated to the study of center manifolds. In Chapter 8 we extend the approach in Chapter 6 to nonuniform exponential trichotomies, and we establish the existence of center manifolds that are as smooth as the vector field. In particular, we obtain simultaneously the existence and smoothness of the center manifolds using a single fixed point problem. In Chapter 9 we show that some symmetries of the differential equations descend to the center manifolds. More precisely, we consider the properties of reversibility and equivariance in time, and we show that the dynamics on the center manifold is reversible or equivariant if the dynamics in the ambient space has the same property.

In the fourth part of the book we study the so-called regularity theory of Lyapunov and its applications to the stability theory of differential equations. We note that this approach is distinct from what is usually called Lyapunov's second method, which is based on the use of Lyapunov functions. In Chapter 10 we provide a detailed exposition of the regularity theory, organized in a pragmatic manner so that it can be used in the last two chapters of the book. In Chapter 11 we extend the regularity theory to the infinite-dimensional setting of Hilbert spaces. Chapter 12 is dedicated to the study of the stability of nonautonomous differential equations using the regularity theory. We note that the notion of Lyapunov regularity is much less restrictive than the notion of uniform stability, and thus we obtain the persistence of the stability of solutions of nonautonomous differential equations under much weaker assumptions.

We are grateful to several people who have helped us in various ways. We particularly would like to thank Jack Hale, Luis Magalhães, Waldyr Oliva, and Carlos Rocha for their support and encouragement along the years as well as their helpful comments on several aspects of our work. We also would like to thank the referees for the careful reading of the manuscript.

We were supported by the Center for Mathematical Analysis, Geometry, and Dynamical Systems, and through Fundação para a Ciência e a Tecnologia by the Programs POCTI/FEDER, POSI and POCI 2010/Fundo Social Europeu, and the grants SFRH/BPD/14404/2003 and SFRH/BPD/26465/2006.

Luis Barreira and Claudia Valls
Lisbon, October 2006

Contents

1

Introduction

In the theory of differential equations, the notion of (uniform) *exponential dichotomy*, introduced by Perron in [69], plays a central role in the study of stable and unstable invariant manifolds. In particular, consider a solution $u(t)$ of the equation $u' = F(u)$ for some differentiable map F in a Banach space. Setting $A(t) = d_{u(t)}F$, the existence of an exponential dichotomy for the linear variational equation

$$v' = A(t)v \tag{1.1}$$

implies the existence of stable and unstable invariant manifolds for the solution $u(t)$, up to mild additional assumptions on the nonlinear part of the vector field. The theory of exponential dichotomies and its applications are well developed. In particular, there exist large classes of linear differential equations with exponential dichotomies. For example, Sacker and Sell [83, 84, 85, 82, 86] discuss sufficient conditions for the existence of exponential dichotomies, also in the infinite-dimensional setting. In a different direction, for geodesic flows on compact smooth Riemannian manifolds with strictly negative sectional curvature, the unit tangent bundle is a hyperbolic set, that is, they are Anosov flows. Furthermore, time changes and small C^1 perturbations of flows with a hyperbolic set also have a hyperbolic set (see for example [49] for details). We refer to the books [24, 41, 46, 88] for details and further references related to exponential dichotomies. We particularly recommend [24] for historical comments. The interested reader may also consult the books [32, 33, 60]. On the other hand, the notion of exponential dichotomy substantially restricts the dynamics and it is important to look for more general types of hyperbolic behavior.

Our main objective is to consider the more general notion of *nonuniform exponential dichotomy* and study in a systematic manner some of its consequences, in particular concerning the existence and smoothness of invariant manifolds for nonautonomous differential equations. Also in the nonuniform setting, we obtain a version of the Grobman–Hartman theorem, the existence of center manifolds, as well as their reversibility and equivariance properties, and an infinite-dimensional version of Lyapunov's regularity theory with applications to the stability of solutions of nonautonomous equations. In comparison with the classical notion of (uniform) exponential dichotomy, the existence of a nonuniform exponential dichotomy is a much weaker hypothesis. In fact, perhaps surprisingly, essentially *any* linear equation as in (1.1), with global solutions and with at least one negative Lyapunov exponent, has a nonuniform exponential dichotomy (see Chapter 10 for details). We emphasize that we always consider nonautonomous differential equations, and with the exception of Chapters 5 and 10 the theory is systematically developed in infinite-dimensional spaces. Another aspect of our approach is that we deal directly with flows or semiflows instead of using their time-1 maps (with the single exception of Chapter 7, where we establish a version of the Grobman–Hartman theorem). Our work is also a contribution to the theory of nonuniformly hyperbolic dynamics (we refer to [1, 2, 3] for detailed expositions of the theory).

We discuss in this chapter the main ideas and motivations behind the theory developed in the book. We also highlight some of the main results and the relations with former work. We mostly follow the order in which the material is presented in the book.

1.1 Exponential contractions

In order to describe the differences between the notions of uniform exponential dichotomy and nonuniform exponential dichotomy, we first consider the case when only contraction is present. We could replace contraction by expansion simply by reversing the time.

Consider a continuous function $t \mapsto A(t)$ with values in the $n \times n$ real matrices for $t \geq 0$. We assume that all solutions of (1.1) are global in the future, that is, are defined for every $t \geq 0$. Let $U(t, s)$ be the evolution operator associated with equation (1.1). This is the operator satisfying

$$U(t, s)v(s) = v(t)$$

for every solution $v(t)$ of (1.1) and every $t \geq s$. We assume in this section that all Lyapunov exponents of solutions of equation (1.1) are negative, that is,

$$\limsup_{t \to +\infty} \frac{1}{t} \log\|v(t)\| < 0 \text{ for each solution } v(t) \text{ of } (1.1). \tag{1.2}$$

We say that $U(t, s)$ is a *(uniform) exponential contraction* if there exist constants a, $c > 0$ such that

$$\|U(t, s)\| \leq ce^{-a(t-s)} \text{ for every } t \geq s.$$

We say that $U(t, s)$ is a *nonuniform exponential contraction* if there exist constants a, $c > 0$ and $b \geq 0$ such that

$$\|U(t,s)\| \le ce^{-a(t-s)+bs} \text{ for every } t \ge s. \tag{1.3}$$

Thus, a nonuniform exponential contraction allows a "spoiling" of the uniform contraction along each solution as the initial time s increases: while the uniform contraction (given by a) is still present in (1.3), and is independent of the initial time $s \ge 0$, we may have the additional exponential term e^{bs} (and thus the nonuniformity along the solution). This means that even though in both cases we have the exponential stability of solutions (due to (1.2)), in the nonuniform case, in order that a given solution is in a prescribed neighborhood, the size of the initial condition may depend on s (while in the uniform case the size can be chosen independently of s).

The following statement is a simple consequence of Theorem 10.6 (the proof of which is inspired in related work in [1]).

Theorem 1.1. *If the equation* (1.1) *satisfies the condition* (1.2)*, then the associated evolution operator $U(t,s)$ is a nonuniform exponential contraction, for which the constant a is any positive number satisfying*

$$a < -\sup_{v_0 \in \mathbb{R}^n} \limsup_{t \to +\infty} \frac{1}{t} \log\|v(t)\|, \tag{1.4}$$

where $v(t)$ is the unique solution of the equation (1.1) *with* $v(0) = v_0$.

We note that the right-hand side of (1.4) is indeed positive (since the lim sup in (1.2) can only take a finite number of values; see Section 10.1). In view of Theorem 1.1, the notion of nonuniform exponential contraction is in fact as weak as possible, since all (exponentially stable) linear equations originate an evolution operator having such a contraction. A similar behavior occurs in the case of nonuniform exponential dichotomies (see Theorem 10.6). Thus, in specific applications we never need to assume the existence of a nonuniform exponential contraction (since this follows from (1.2)) but instead we look for conditions on a and b which ensure the desired results. For example, in general we are only able to establish the stability of the zero solution of (1.1) under sufficiently small perturbations provided that b/a is sufficiently small (see Chapter 12 for related results).

In view of this discussion it is also important to give a sharp estimate for b. We refer to Section 10.3 for details; here, we consider only the case of triangular matrices. The following statement is a simple consequence of Theorems 10.6 and 10.8.

Theorem 1.2. *If the matrix $A(t)$ is upper triangular for every $t \ge 0$, then the constant b can be any number satisfying*

$$b > \sum_{k=1}^{n} \left(\limsup_{t \to +\infty} \frac{1}{t} \int_0^t a_k(\tau)\, d\tau - \liminf_{t \to +\infty} \frac{1}{t} \int_0^t a_k(\tau)\, d\tau \right),$$

where $a_1(t), \ldots, a_n(t)$ are the entries in the diagonal of $A(t)$.

See Chapter 11 for generalizations of Theorems 1.1 and 1.2 to infinite-dimensional spaces.

1.2 Exponential dichotomies and stable manifolds

We now consider the more general case of nonuniform exponential dichotomies. These are composed of nonuniform contractions and nonuniform expansions (see Section 1.1). We also present a first consequence of the existence of an exponential dichotomy, namely the existence of invariant stable manifolds for any sufficiently small perturbation.

Consider a Banach space X and a continuous function $t \mapsto A(t)$ such that $A(t)$ is a bounded linear operator on X for each $t \geq 0$. We assume again that all solutions of (1.1) are global in the future, that is, are defined for every $t \geq 0$. Let $T(t,s)$ be the evolution operator associated with equation (1.1). This is the operator satisfying

$$T(t,s)v(s) = v(t)$$

for every solution $v(t)$ of (1.1) and every $t \geq s$. For simplicity of the exposition, we assume that the evolution operator $T(t,s)$ has a decomposition in block form

$$T(t,s) = (U(t,s), V(t,s))$$

into evolution operators with respect to some invariant decomposition $X = E \oplus F$ (which is independent of the time t). We emphasize that in the remaining chapters we do not assume that $T(t,s)$ has a decomposition in block form.

We say that the equation (1.1) admits a *nonuniform exponential dichotomy* if there exist constants $\lambda < 0 \leq \mu$ and a, b, $K > 0$, such that for every $t \geq s \geq 0$,

$$\|U(t,s)\| \leq Ke^{\lambda(t-s)+as} \quad \text{and} \quad \|V(t,s)^{-1}\| \leq Ke^{-\mu(t-s)+bt}. \tag{1.5}$$

The constants λ and μ play the role of Lyapunov exponents, while a and b measure the nonuniformity of the dichotomy. The assumption $\lambda < 0$ means that there is at least one negative Lyapunov exponent.

We now consider the equation

$$v' = A(t)v + f(t,v), \tag{1.6}$$

where the perturbation $f(t,v)$ is a continuous function defined for $t \geq 0$ and $v \in X$, such that $f(t,0) = 0$ for every $t \geq 0$ (and thus the origin is also a solution of (1.6)).

The following is one of our main results on the existence of stable manifolds for a nonautonomous differential equation, and is an immediate consequence of Theorem 4.1.

Theorem 1.3. *Assume that the equation* (1.1) *admits a nonuniform exponential dichotomy, and that there exist* $c > 0$ *and* $q > 0$ *such that*

$$\|f(t,u) - f(t,v)\| \leq c\|u - v\|(\|u\|^q + \|v\|^q)$$

for every $t \geq 0$ *and* $u, v \in X$. *If*

$$\lambda + a + (a+b)/q < 0 \quad \text{and} \quad \lambda + b < \mu, \tag{1.7}$$

then there exists a Lipschitz function $\varphi\colon U \to F$, *where* $U \subset \mathbb{R}_0^+ \times E$ *is an open neighborhood of the line* $\mathbb{R}_0^+ \times \{0\}$, *such that its graph* $\mathcal{W} \subset \mathbb{R} \times X$ *has the following properties:*

1. $(t, 0) \in \mathcal{W}$ *for every* $t \geq 0$;
2. $\mathcal{W}$ *is forward invariant under the semiflow* Ψ_τ *on* $\mathbb{R}_0^+ \times X$ *generated by the autonomous system*
$$t' = 1, \quad v' = A(t)v + f(t, v);$$
3. *there exists* $D > 0$ *such that for every* (s, u), $(s, v) \in \mathcal{W}$ *and* $\tau \geq 0$, *we have*
$$\|\Psi_\tau(s, u) - \Psi_\tau(s, v)\| \leq De^{\lambda\tau + as}\|u - v\|.$$

We refer to Section 4.2 for a detailed formulation. We observe that the Lipschitz invariant manifolds constructed in Theorem 1.3 are in fact as smooth as the vector filed. We refer to Chapters 5 and 6 for details.

Note that the first inequality in (1.7) is satisfied for a given $a < |\lambda|$ provided that q, the order of the perturbation, is sufficiently large. Furthermore, both inequalities in (1.7) are automatically satisfied when a and b are sufficiently small. The "small" exponentials e^{as} and e^{bt} in (1.5), that are not present in the case of a uniform exponential dichotomy, are the main cause of difficulties. On the other hand, it turns out that the smallness of the nonuniformity is a rather common phenomenon from the point of view of ergodic theory: almost all linear variational equations obtained from a measure-preserving flow on a smooth Riemannian manifold admit a nonuniform exponential dichotomy with arbitrarily small nonuniformity (see Theorem 10.6).

Our definition of weak nonuniform exponential dichotomy in (1.5) is inspired in the notion of uniform exponential dichotomy and in the notion of nonuniformly hyperbolic trajectory (see Sections 4.3 and 5.2). Our work is also a contribution to the theory of nonuniformly hyperbolic dynamics. We refer to [1, 3] for detailed expositions of parts of the theory and to the survey [2] for a detailed description of its contemporary status. The theory goes back to the landmark works of Oseledets [65] and Pesin [70, 71, 72]. Since then it became an important part of the general theory of dynamical systems and a principal tool in the study of stochastic behavior. We note that the nonuniform hyperbolicity conditions can be expressed in terms of the Lyapunov exponents. For example, almost all trajectories of a dynamical system preserving a finite invariant measure with nonzero Lyapunov exponents are nonuniformly hyperbolic.

Among the most important properties due to nonuniform hyperbolicity is the existence of stable and unstable manifolds, and their absolute continuity property established by Pesin in [70]. The theory also describes the ergodic properties of dynamical systems with a finite invariant measure absolutely continuous with respect to the volume [71], and expresses the Kolmogorov–Sinai entropy in terms of the Lyapunov exponents by the Pesin entropy formula [71] (see also [55]). In another direction, combining the nonuniform hyperbolicity with the nontrivial recurrence guaranteed by the existence of a finite invariant measure, the fundamental work of Katok [48] revealed a very rich and complicated orbit structure, including an exponential growth rate for the number of periodic points measured by the topological entropy, and an approximation by uniformly hyperbolic horseshoes of the entropy of an invariant measure (see also [50]).

Here we concentrate our attention on the stable manifold theorem. We first briefly describe the relevant references. The proof by Pesin in [70] is an elaboration of the classical work of Perron. His approach was extended by Katok and Strelcyn in [51] for maps with singularities. In [80], Ruelle obtained a proof of the stable manifold theorem based on the study of perturbations of products of matrices in Oseledets' multiplicative ergodic theorem [65]. Another proof is due to Pugh and Shub in [78] with an elaboration of the classical work of Hadamard using graph transform techniques. In [37] Fathi, Herman and Yoccoz provided a detailed exposition of the stable manifold theorem essentially following the approaches of Pesin and Ruelle. We refer to [3] for further details. There exist also versions of the stable manifold theorem for dynamical systems in infinite-dimensional spaces. In [81] Ruelle established a corresponding version in Hilbert spaces, following his approach in [80]. In [58] Mañé considered transformations in Banach spaces under some compactness and invertibility assumptions, including the case of differentiable maps with compact derivative at each point. The results of Mañé were extended by Thieullen in [92] for a class of transformations satisfying a certain asymptotic compactness. We refer the reader to the book [42] for a detailed discussion of the geometric theory of dynamical systems in infinite-dimensional spaces.

We note that in the above works the dynamics is assumed to be of class $C^{1+\varepsilon}$ for some $\varepsilon > 0$. On the other hand, in [77] Pugh constructed a C^1 diffeomorphism in a manifold of dimension 4, that is not of class $C^{1+\varepsilon}$ for any ε, and for which there exists no invariant manifold tangent to a given stable space such that the trajectories along the invariant manifold travel with exponential speed. We refer to [3] for a detailed description of the diffeomorphism. Nevertheless, although this example shows that the hypothesis $\varepsilon > 0$ is crucial in the stable manifold theorem it does not forbid the existence of families of C^1 dynamics which are not of class $C^{1+\varepsilon}$ for any ε but for which there still exist stable manifolds. Indeed, Theorem 5.1 implies the existence of invariant stable manifolds for the nonuniformly hyperbolic trajectories of a large family of maps that, in general, are *at most* of class C^1. A detailed presentation is given in Section 5.3.

There are some differences between our approach and the usual approach in the theory of nonuniformly hyperbolic dynamics. In particular, we start from a linear equation $v' = A(t)v$ instead of a linear variational equation