Theta Functions Bowdoin 1987

Volume 49–Part 1

PROCEEDINGS OF SYMPOSIA IN PURE MATHEMATICS

AMERICAN MATHEMATICAL SOCIETY

PROCEEDINGS OF SYMPOSIA
IN PURE MATHEMATICS
Volume 49, Part 1

Theta Functions
Bowdoin 1987

Leon Ehrenpreis and
Robert C. Gunning, Editors

AMERICAN MATHEMATICAL SOCIETY
PROVIDENCE, RHODE ISLAND

PROCEEDINGS OF THE SUMMER RESEARCH INSTITUTE
ON THETA FUNCTIONS
HELD AT BOWDOIN COLLEGE
BRUNSWICK, MAINE
JULY 6–24, 1987

with support from the National Science Foundation, Grant DMS–8611435

1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 00A11.

Library of Congress Cataloging-in-Publication Data

Theta functions, Bowdoin 1987/Leon Ehrenpreis and Robert C. Gunning, editors.
p. cm. – (Proceedings of symposia in pure mathematics, ISSN 0082-0717; v. 49)
Proceedings of the thirty fifth Summer Research Institute on Theta Functions held at Bowdoin College, Brunswick, Me., July 6–24, 1987; sponsored by the American Mathematical Society.
Includes bibliographies.
1. Functions, Theta–Congresses. I. Ehrenpreis, Leon. II. Gunning, R. C. (Robert Clifford), 1931- . III. Summer Research Institute on Theta Functions (35th: 1987: Brunswick, Me.) IV. American Mathematical Society. V. Series.
QA345.T47 1989 515′.984–dc20 89-6723
ISBN 0-8218-1483-4 (part 1) CIP
ISBN 0-8218-1484-2 (part 2)
ISBN 0-8218-1485-0 (set) (alk. paper)

The paper used in this book is acid-free and falls within the guidelines
established to ensure permanence and durability. ♾
This publication was typeset using $\mathcal{A}_{\mathcal{M}}\mathcal{S}$-TEX,
the American Mathematical Society's TEX macro system.

Preface

Theta functions apparently first appeared in the forms $\sum_{n=0}^{\infty} m^{n^2}$, $\sum_{n=0}^{\infty} m^{1/2n(n+1)}$, $\sum_{n=0}^{\infty} m^{1/2n(n+3)}$ in the work of Jakob Bernoulli. In his work on partition theory, Euler introduced a second variable ζ and studied functions of the form $\prod_{n=1}^{\infty}(1 - q^n\zeta)^{-1}$. For Euler, the primary objects were partition functions such as $\prod(1 - q^n)$, but the function $\prod(1 - q^n\zeta)^{-1}$ was considered as a function of ζ with q occurring as a parameter; after deriving identities for the function of ζ he then set $\zeta = 1$.

Jacobi made two important notational changes that turned out to be crucial for the modern development. He replaced q by $e^{\pi i\tau}$ and ζ by e^{2iz}; thus was born the theta function in its present form

$$\theta(\tau, z) = \sum e^{\pi i n^2\tau + 2inz}.$$

The change from q to τ allowed him to formulate the "imaginary transformation" $\tau \to -1/\tau$, which together with the obvious transformation $\tau \to \tau + 2$ leads to the modular group and eventually to the modern theory of modular forms and their ramifications. (The formulation of the modular group in the variable q is complicated; see the paper by Ehrenpreis in this volume.)

In addition, Jacobi studied $\theta(\tau, z)$ as a function of z in its own right. The quasi double periodicity under $z \to z + \pi$ and $z \to z + \pi\tau$ enabled him to relate theta functions as functions of z to elliptic function theory. For Jacobi as for Euler the primary working variable was z. Of course, this theory has had far reaching generalizations to higher genera Riemann surfaces, abelian varieties, etc.

Surprisingly, theta functions made their appearance in another case of nineteenth century mathematics, namely mechanics. It was discovered by Carl Neumann and Jacobi that certain mechanical (Hamiltonian) systems could be explicitly integrated by means of theta functions. These ideas could have formed the foundation of some of the modern ideas on KdV, KP, and integrable systems in general, but the modern viewpoint seems to have been discovered without knowledge of the eighteenth century results.

When the organizing committee met to discuss the possibility of a conference on theta functions, we saw how perfectly the notation $\theta(\tau, z)$ fit into a three week conference: one week for τ, one week for z, and one week for the comma. (This conforms to the above described three aspects of theta functions that appeared in the nineteenth century.) The conference was thereby organized accordingly. The first week was devoted to the comma, that is, to the interplay of τ and z. The sections on infinite analysis, integrable systems, Kac-Moody algebras, lattice models, and physics are, roughly speaking, devoted to this interplay; the sections on Jacobi varieties, Prym varieties, and algebraic geometry emphasize the z variable. These sections form Part 1 of Volume 49. The sections on modular forms, number theory, and combinatorics emphasize the τ variable. They comprise Part 2 of Volume 49.

It was our hope in organizing the conference that the presentation of a cross section of modern work on theta functions would enable mathematicians to see where we stand now and in what directions we should go in the future.

Leon Ehrenpreis
Robert C. Gunning

Contents

Kac-Moody Algebras

Lattice Models

Physics

Jacobian Varieties

Prym Varieties

Algebraic Geometry

PART 2

Modular Forms

Number Theory

Combinatorics

Infinite Analysis

Proceedings of Symposia in Pure Mathematics
Volume **49** (1989), Part 1

Systems of Linear Differential Equations of Infinite Order: An Aspect of Infinite Analysis

TAKAHIRO KAWAI

1. Introduction.

1.1. Although several years have elapsed since Professor M. Sato coined the terminology "infinite analysis," its exact content is not yet fixed. It is, however, commonly accepted that the analysis of theta functions should be its prototype. The purpose of this article is to describe one aspect of the infinite analysis related to theta functions, i.e., analyzing the theta zero-value by a system of linear differential equations of infinite order that it satisfies. Another aspect of infinite analysis related to KP-hierarchy will be explained by Sato's contribution to this volume.

Let us begin our discussion by showing how linear differential operators of infinite order in our sense appear in analysis. For the sake of simplicity, we confine the discussion to the one-dimensional case in this subsection. Note that a linear differential operator of infinite order is not an arbitrary sum of infinitely many differential operators.

Let 0 be the origin of $\mathbb{R} \subset \mathbb{C}$, and let $\mathscr{B}_{\{0\}}$ and $\mathscr{D}\!\mathscr{b}_{\{0\}}$ respectively denote the space of hyperfunctions supported at 0 and that of distributions supported at 0. Then it is known that

(1.1.1) $$\mathscr{B}_{\{0\}} = \mathscr{O}(\mathbb{C}\backslash\{0\})/\mathscr{O}(\mathbb{C}),$$

and that

(1.1.2) a representative of Dirac's δ-function $\delta(x)$ in $\mathscr{O}(\mathbb{C}\backslash\{0\})$ is given by $-1/2\pi i z$.

It is also well known that

(1.1.3) $\mathscr{D}\!\mathscr{b}_{\{0\}}$ consists of distributions of the form $P(D_x)\delta(x)$, where $P(\zeta) = \sum_{n=0}^{m} a_n \zeta^n$ is a polynomial of $\zeta \in \mathbb{C}$.

1980 *Mathematics Subject Classification* (1985 *Revision*). Secondary 35N99, 46F15, 58G07.

Hence a distribution with its support at 0 has a representative of the form

$$-\frac{1}{2\pi i}\left(\sum_{n=0}^{m} a_n \frac{(-1)^n n!}{z^{n+1}}\right) \tag{1.1.4}$$

in $\mathscr{O}(\mathbb{C}\setminus\{0\})$; in particular, it has only a pole at 0. On the other hand, an element in $\mathscr{O}(\mathbb{C}\setminus\{0\})$ has the form

$$\sum_{j=1}^{\infty} f_j z^{-j} + \sum_{k=0}^{\infty} g_k z^k \tag{1.1.5}$$

with

$$\lim \sqrt[j]{|f_j|} = 0 \tag{1.1.6}$$

and $\sum_{k=0}^{\infty} g_k z^k$ being in $\mathscr{O}(\mathbb{C})$. We may rewrite $\sum_{j=1}^{\infty} f_j z^{-j}$ in the form

$$\sum_{j=1}^{\infty} f_j \frac{(-1)^{j-1}}{(j-1)!} D^{j-1}\left(\frac{1}{z}\right). \tag{1.1.7}$$

Now let $J(\zeta)$ denote the following infinite series:

$$\sum_{j=0}^{\infty} f_{j+1} \frac{(-1)^j}{j!} \zeta^j. \tag{1.1.8}$$

Then it follows from (1.1.6) that

(1.1.9) $J(\zeta)$ is an entire function of ζ

and further that

(1.1.10) $J(\zeta)$ is of order 1 with minimal type, i.e., $|J(\zeta)| \leq A_\varepsilon \exp(\varepsilon|\zeta|)$ for any $\varepsilon > 0$.

Actually (1.1.10) is equivalent to the following condition (1.1.11) on the coefficients of the Taylor expansion $\sum_{j=0}^{\infty} h_j \zeta^j$ of an entire function $J(\zeta)$:

(1.1.11) For each $\varepsilon > 0$, there exists a constant C_ε such that $|h_j| \leq C_\varepsilon \varepsilon^j / j!$ holds for every j in $\mathbb{N}$.

For such an entire function $J(\zeta)$ and a holomorphic function $\phi(z)$ defined near $z = z_0$, we can define $J(D_z)\phi(z)$ to be $\sum_{j=0}^{\infty} h_j D_z^j \phi(z)$, which is again holomorphic near $z = z_0$. We call an operator $J(D_z)$ thus defined a linear differential operator of infinite order and with constant coefficients. If we replace the constant h_j by a holomorphic function $h_j(z)$ defined on an open neighborhood ω of z_0 (independent of j) such that $\sup_\omega |h_j(z)|$ is dominated by $C_\varepsilon \varepsilon^j / j!$ for each $\varepsilon > 0$, then we get a general linear differential operator of infinite order (defined on ω). One important property of such an operator is that it has a local property in that it acts on the sheaf $\mathscr{O}$ of holomorphic functions as a sheaf homomorphism. This property guarantees that it also acts on the sheaf $\mathscr{B}$ of hyperfunctions and the sheaf $\mathscr{C}$ of microfunctions as sheaf homomorphisms. [See [**K**^{3}] for the definition of these sheaves.] Note, however, that it does not act on the sheaf $\mathscr{D}\mathscr{b}$ of distributions, or even on the sheaf of infinitely differentiable functions.

We know that linear differential operators of infinite order are intrinsically defined on a (real) analytic manifold, and that they constitute a sheaf. The sheaf is denoted by $\mathscr{D}^\infty$.

What has been observed so far may be summarized in a symbolic manner as follows:

$$\begin{aligned}(1.1.12)\qquad &\mathscr{B}_{\{0\}} : \mathscr{D}\mathscr{b}_{\{0\}}\\ &= \{\text{essential singularities at } 0\} : \{\text{pole singularities at } 0\}\\ &= \{\text{entire functions satisfying } (1.1.10)\} : \{\text{polynomials}\}\\ &= \{\text{linear differential operators of infinite order and with}\\ &\quad \text{constant coefficients}\} : \{\text{linear differential operators of}\\ &\quad \text{finite order and with constant coefficients}\}.\end{aligned}$$

See [$\mathbf{K}^3$, p. 151] for the results in the higher-dimensional case. Note also that the second equality can be understood as the Borel transformation.

1.2. The simplest example of a linear differential operator of really infinite order is, probably,

$$(1.2.1)\qquad \cosh(a\sqrt{D_z}) = \sum_{n=0}^{\infty} \frac{a^{2n}D_z^n}{(2n)!} \qquad (a \in \mathbb{C}).$$

One of the most important discoveries in [**SKK**] is that operators of this sort are not an object of curiosity. Actually the so-called structure theorem for systems of (micro)differential equations obtained in [**SKK**] requires the essential use of operators of infinite order. Instead of going into the details of this topic, we explain how they appear in a concrete example. This example is, as might be easily surmised, closely tied up with the analysis of the theta zero-value $\sum_{\nu\in\mathbf{Z}} \exp(\pi i\nu^2 t)$.

Let us now consider the following equation on $\mathbb{C}^2_{(x,t)}$:

$$(1.2.2)\qquad \begin{pmatrix}\partial_x & 0\\ 0 & \partial_x\end{pmatrix}\begin{pmatrix}u\\ v\end{pmatrix} = Q_c(\partial_t)\begin{pmatrix}u\\ v\end{pmatrix},$$

where

$$(1.2.3)\qquad Q_c(\partial_t) = \begin{pmatrix}0 & 1\\ c\partial_t & 0\end{pmatrix} \qquad (c \in \mathbb{C})$$

and $\partial_x = \partial/\partial x$, $\partial_t = \partial/\partial t$. Needless to say, (1.2.2) is equivalent to

$$(1.2.4)\qquad \begin{cases} \partial_x^2 u = c\partial_t u,\\ v = \partial_x u.\end{cases}$$

Throughout this subsection, we fix the constant c ($\neq 0$) and omit the subscript c in Q_c in what follows:

Set

$$(1.2.5)\qquad W = \exp(xQ) \quad \left(= \sum_{n=0}^{\infty} \frac{(xQ)^n}{n!}\right).$$

Then, using the relation

$$Q^2 = \begin{pmatrix} c\partial_t & 0 \\ 0 & c\partial_t \end{pmatrix}, \tag{1.2.6}$$

we find

$$W = \sum_{n=0}^{\infty} \frac{c^n x^{2n}}{(2n)!} \partial_t^n \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \sum_{n=0}^{\infty} \frac{c^n x^{2n+1}}{(2n+1)!} \partial_t^n \begin{pmatrix} 0 & 1 \\ c\partial_t & 0 \end{pmatrix}. \tag{1.2.7}$$

Then it is clear that each component of the matrix W is a linear differential operator of infinite order (cf. (1.2.1)). It is also obvious that W is invertible, because

$$\exp(xQ)\exp(-xQ) = I. \tag{1.2.8}$$

Further, we can easily verify

$$W^{-1}\left(\begin{pmatrix} \partial_x & 0 \\ 0 & \partial_x \end{pmatrix} - Q\right) W = \begin{pmatrix} \partial_x & 0 \\ 0 & \partial_x \end{pmatrix}. \tag{1.2.9}$$

Thus the equation (1.2.2) has been transformed into the direct sum of the simplest equations of the sort, i.e., $\partial_x u = 0$. This clear-cut result is obtained only with the aid of operators of infinite order; it is impossible to obtain such a result, if we use only operators of finite order.

1.3. Although it is a digression from our purpose, we mention another interesting example which shows the usefulness of linear differential operators of infinite order.

Let us consider the following ordinary differential equation with irregular singularities:

$$(x^2 D - a)U = 0 \qquad (a \in \mathbb{C}\setminus\{0\}). \tag{1.3.1}$$

Then this equation is equivalent to the following equation with regular singularities, if transformations by linear differential operators of infinite order are allowed:

$$\begin{pmatrix} x & -a \\ 0 & xD \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = 0. \tag{1.3.2}$$

The transformation is concretely given as follows:

$$\begin{pmatrix} u \\ -xDu \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{aD}} I_1(2\sqrt{aD}) & -2\sqrt{aD}K_1^*(2\sqrt{aD}) \\ I_0(2\sqrt{aD}) & 2aDK_0^*(2\sqrt{aD}) \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}, \tag{1.3.3}$$

$$\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} 2aDK_0^*(2\sqrt{aD}) & 2\sqrt{aD}K_1^*(2\sqrt{aD}) \\ -I_0(2\sqrt{aD}) & \frac{1}{\sqrt{aD}} I_1(2\sqrt{aD}) \end{pmatrix} \begin{pmatrix} u \\ -xDu \end{pmatrix}. \tag{1.3.4}$$

Here

$$I_\nu(\zeta) = \left(\frac{\zeta}{2}\right)^\nu \left(\sum_{n=0}^{\infty} \frac{(\zeta/2)^{2n}}{n!\Gamma(\nu+n+1)}\right)$$

and

$$\begin{aligned} K_n^*(\zeta) &= (-1)^n I_n(\zeta)\log\left(\frac{\zeta}{2}\right) + K_n(\zeta) \\ &= \frac{(-1)^n}{2}\left(\sum_{k=0}^{\infty} \frac{\psi(k+1)+\psi(k+n+1)}{k!(n+k)!}\left(\frac{\zeta}{2}\right)^{n+2k}\right) \\ &\quad + \frac{1}{2}\left(\sum_{r=0}^{n-1}(-1)^r \frac{(n-r-1)}{r!}\left(\frac{\zeta}{2}\right)^{2r-n}\right), \end{aligned}$$

where

$$\psi(n) = \sum_{k=1}^{n-1}\frac{1}{k} - \gamma \qquad (\gamma\text{: the Euler constant}).$$

This phenomenon is most thoroughly studied for holonomic systems of microdifferential equations by **[KK]**. See **[M]** and **[U]** for related topics.

2. Linear differential equations of infinite order.

2.1. In §1 we have shown that linear differential *operators* of infinite order are natural and important in analysis. So far, however, we have not discussed linear differential *equations* of infinite order. It was Sato **[S]** who first observed the importance of such equations in analyzing transcendental functions such as theta zero-value. In the next subsection, we will give the explicit form of such equations in the case of the theta zero-value $\vartheta(t) = \sum_\nu \exp(\pi i \nu^2 t)$.

2.2. Let P and Q respectively denote the matrix of linear differential operators of finite order given as follows:

$$P = \begin{pmatrix} 0 & t \\ 4\pi i(t\partial_t + 1/2) & 0 \end{pmatrix}, \tag{2.2.1}$$

$$Q = \begin{pmatrix} 0 & 1 \\ 4\pi i\partial_t & 0 \end{pmatrix}. \tag{2.2.2}$$

Let Φ and Ψ respectively denote $\exp P - I$ $(= \sum_{n=1}^{\infty} P^n/n!)$ and $\exp Q - I$ $(= \sum_{n=1}^{\infty} Q^n/n!)$. Then one can verify that both Φ and Ψ are matrices whose entries are linear differential operators of infinite order (cf. §1.2). Furthermore, using the commutation relation

$$[Q, P] = 2\pi i I_2, \tag{2.2.3}$$

one can verify

$$\Phi\Psi = \Psi\Phi. \tag{2.2.4}$$