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FUNDAMENTALS OF QUEUEING THEORY

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continued on back

FUNDAMENTALS OF QUEUEING THEORY

A WILEY PUBLICATION IN APPLIED STATISTICS

To
[2(Alice) + Paul]

PREFACE

This book is the outgrowth of our experiences in teaching a one-semester course in queueing theory, which serves as an elective in a Master of Science operations research program offered by the Department of Operations Research, School of Engineering and Applied Science of The George Washington University. We believe there has been a great gap in books available which could serve as a text for such a course. In writing this book, we have expanded, modified, and revised our course notes several times, and also incorporated, where applicable, our experiences in the research and application of queueing theory.

COURSE COVERAGE

This book can be used as a text for a one or two semester, or a one or two quarter course, such as the one described above, namely, a graduate-level elective course in an operations research, systems analysis, or industrial engineering program of an engineering school or a quantitatively oriented program of a business school. The chart below is our recommendation for the material to include in courses of varying duration. Naturally, the individual instructor should feel free to make adjustments depending on his or her personal interests and the background of the students.

In addition, we feel that this book can also be extremely valuable for analysts working in industry or government who may wish to fill in some theoretical gaps in the development of queueing models. The last two chapters on simulation and application, respectively, might be of particular interest to practicing analysts. In toto, we think that the book will serve as a comprehensive reference in the area of queueing theory. It has a degree of mathematical sophistication which will allow the understanding and use

Recommended Coverage

Course Length	Text Material
One Quarter	Chapters 1, 2 (through 2.6.1), 3 (omit 3.2.1 and 3.5.2), 4 (through 4.4.1), 5 (5.1.1 only)
One Semester	Chapters 1, 2, 3, 4, 5 (5.1.1, 5.1.2, 5.3.1), 6 (6.1 only), 7 (7.2 only)
Two Quarters	Chapters 1, 2, 3, 4, 5 (5.1.1, 5.1.2, 5.1.10, 5.2.2, 5.3), 6 (6.1 only), 7 (7.2 only), 8, 9
Two Semesters	Entire Text

of most of the important queueing models, but is not so mathematically abstract and imposing as to prevent all but mathematicians from understanding the material.

PREREQUISITE KNOWLEDGE

The background assumed on the part of the reader is a knowledge of undergraduate differential and integral calculus and of elements of differential equations, and the experience of a probability and statistics course based upon a calculus foundation. Some knowledge of special "higher" mathematical topics such as transforms, difference equations, and Markov processes would be helpful for the more advanced material, although certainly not necessary, since appendices on these subjects are included. Thus the undergraduate background of most engineering, physical science, and mathematics majors, as well as *some* economics, business administration, and social science majors, would be adequate.

PHILOSOPHY AND ORGANIZATION

The general plan of the book is to present, in detail, the basic material in the early chapters. For example, in Chapter 1 we spend a sizable amount of time on deterministic queues. In addition to adding this topic for completeness, we believe it serves to familiarize the reader in a graphic way with the actual "workings" of a queue, such as, how customers arrive and depart and how the line builds up and depletes.

In Chapter 2 we develop the basic exponential-interarrival-time exponential-service-time single-channel model ($M/M/1$) and then in Chapter 3 derive the general birth–death equations, showing that the $M/M/1$ model falls out as a special case. We purposely do this partial repetition as a pedagogical tool, since we feel that $M/M/1$ more naturally follows from the Poisson process than does the birth–death process. Furthermore, the simplicity of $M/M/1$ allows us to illustrate early in the text the three methods of solving difference equations, namely, iteration, generating functions, and linear difference methods (the latter sometimes referred to as the operator or characteristic-equation approach).

We introduce only touches of transient analyses to show their complexity, but give the reader a feel for the speed of convergence to steady state by observing in detail the very simple exponential-exponential single-channel no-waiting-room ($M/M/1/1$) case.

We also provide an extensive discussion of the concepts of steady state and ergodicity, topics which often have been glossed over or completely ignored. All this is done in Chapter 2, so that the student, after reading the first two chapters, should have a solid feeling for the nature and types of analyses involved in basic queueing models.

The general approach used in this text (except where other types of analyses are indicated) is the rather formal one of first setting-up difference equations, and then obtaining the differential-difference equations and finally the steady-state difference equations which then lead to the steady-state probabilities. We do, however, mention briefly (Problem 2.1) the method of detailed (or stochastic) balance which yields the steady-state probabilities directly, but our personal taste dictated the more formal procedure.

While the book is titled "Fundamentals," some of the material in Chapters 5 and 6 (especially Chapter 6) is quite advanced. We do not go into great detail for this advanced material, but present results, sketch derivations, and reference the literature freely, so that the interested reader can gain greater depth by going to the references if he or she so desires. The first six chapters deal with the development of analytical queueing models while the last three deal with application and implementation.

ACKNOWLEDGMENTS

Many colleagues and students have contributed to this book through their encouragement and day-to-day discussions on the topics in general and portions of the text in particular. We are especially thankful for the help provided by Professors George S. Fishman, Irwin Greenberg, N. U.

Prabhu, and Nozer D. Singpurwalla, and by Dr. Daniel P. Heyman. We are indebted to the administration of The George Washington University, to Dr. Harold Liebowitz, Dean of Engineering and Applied Science, and to Professor William H. Marlow, Chairman, Department of Operations Research and Director, Institute for Management Science and Engineering, for their support of this effort. We are also most grateful to the Office of Naval Research for support of portions of this work under Contract N00014-67-A-0214-0001. Special thanks are due Mr. Ross Tomlinson, Mrs. Bettie Taggart, Mrs, B. J. Walker, Miss Mary L. Vincent and Miss Robin Meader for their invaluable aid in computer programming, typing, and proofreading. Research Assistants T. R. Thiagarajan and K. L. Chhabra provided help in the area of computer programming and calculations. In addition, thanks go to our many students, particularly F. Al-Khayyal, B. Rider, and S. S. Shanker for finding typographical (and other) errors, and providing solutions for many of the problems. We also wish to express sincere gratitude to our many teachers through the years of our education, particularly our advisors, Professors Robert E. Bechhofer, Clifford W. Marshall, Andrew Schultz, Jr., and Lionel Weiss.

Finally, we wish to express our sincere appreciation to our wives for their constant encouragement and reminders that we should complete our task forthwith, so that we might devote more time to our families, and to our parents, Frank and Marion Gross, and Benjamin and Marion Harris, without whose encouragement to complete our education this work would never have been undertaken.

DONALD GROSS
CARL M. HARRIS

Washington, D.C.
January 1974

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CONTENTS

CHAPTER