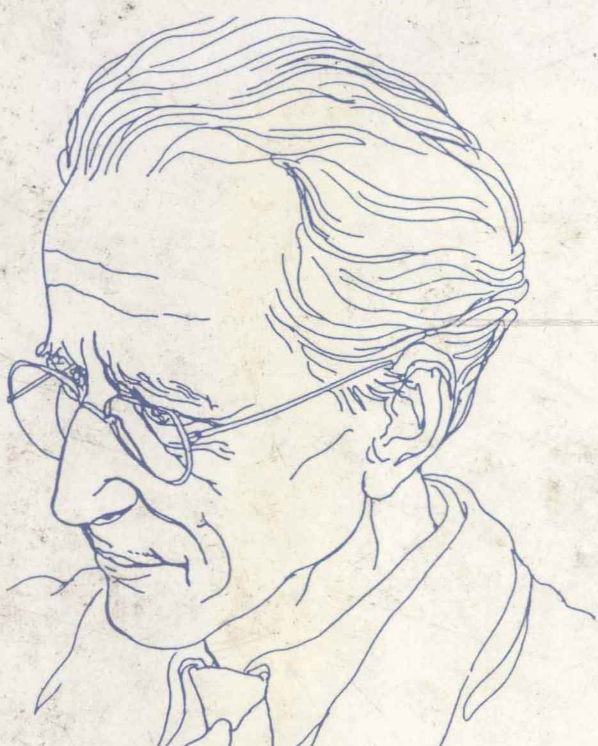
SCHRÖDINGER OPERATORS, STANDARD and NON-STANDARD

Dubna, USSR 6–10 September 1988



Editors: **Pavel Exner and Petr Šeba**

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Preface

One can hardly imagine something more usual in quantum mechanics than the Schroedinger equation ; every textbook starts from it. Nevertheless, at the beginning of its seventh decade it still attracts a vivid interest and stimulates a research work. It has been the main subject of the conference whose summary is presented in this volume.

The development of quantum mechanics is characterized by the growing role of rigorous methods. Already the pioneering works of von Neumann, Kato and others showed that it was advantageous to study the Schroedinger equation as an operator problem in a suitable functional space. This approach has later proven as a very fruitful one and the theory of Schroedinger operators became a standard topic of mathematical physics.

Of course, some people are used to claim that from the physical point of view the non-relativistic quantum mechanics is a (more or less) finished chapter, where one can get new results by better computational methods, but nothing physically surprising. We are convinced that the meetings like ours help to demonstrate they are completely wrong. Just the opposite is true : recent progress in the theory of Schroedinger operators demonstrates that new mathematical methods *and* interesting physical applications go hand by hand.

In this process the notion of a Schroedinger operator itself is gradually broadening. The core of the theory is represented, of course, by the sort of "classical" problems dealing with various deterministic and not very singular potentials. Even here one can still find many unsolved problems, in particular, concerning the spectral and scattering properties of Hamiltonians of atomic and molecular systems. Naturally, a part of the contributions to our conference has been oriented at such problems, *a fortiori*, if one of the pioneers of the Schroedinger operator theory, Professor M.S.Birman, has been among the participants together with some of his disciples.

At the same time, new physical problems and approaches stimulated study of Schroedinger operators with point and contact interactions, i.e., distribution-like potentials, with stochastic potentials, Schroedinger operators on complicated spatial regions, lattice analogues of Schroedinger operators etc. ; we tried to cover this diversity of generalizations in the conference title by the word non-standard. We hope that the present volume provides an up-to-date information about a substantial part of this rapidly growing field.

The Dubna conference was a part of a recently started series of meetings on mathematical physics. Its aim is to use the potential of East-West scientific exchange on a not very official, rather working level. This channel has been nearly closed for a long time ; we hope that it will prove its effectivity.

We want to thank Joint Institute for Nuclear Research for providing conference facilities and Prof.A.Sisakian for his support. We are particularly grateful to Dr.Nguyen Dao Dang who has drawn the cover portrait of Erwin Schroedinger as well as the excellent portraits of our main speakers which accompany their lectures and convey, as we hope, something of the conference atmosphere.

Dubna, February 1989

The editors

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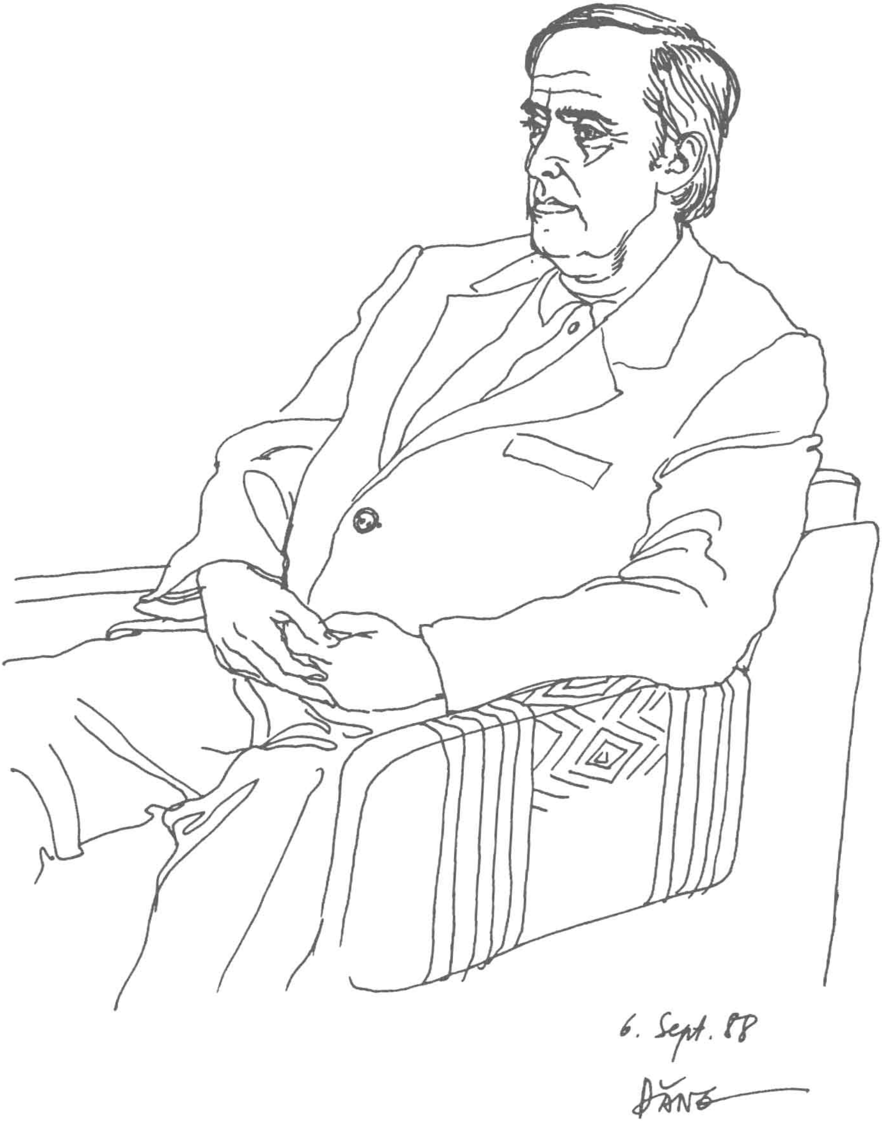
6. Sept. 88

INTERPOLATION ESTIMATES FOR THE NUMBER OF NEGATIVE EIGENVALUES OF A SCHROEDINGER OPERATOR*)

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The problem of estimating the number $N = N(V)$ of negative eigenvalues of a Schroedinger operator from above can be regarded now as a classical one. The results obtained before 1977 are reviewed in the book [1] (cf also the paper [2]); among more recent works we mention the papers [3-6] and the references contained therein**). There is a growing interest to the case of (negative) potentials V which do not fulfill the condition

$$V \in L_{m/2}(\mathbb{R}^m) \quad ; \; m \geq 3$$

(this condition ensures a correct quasiclassical behavior of the numbers $N(\alpha V)$ as $\alpha \to \infty$). We are going to demonstrate that the analysis of the case

$$V \notin L_{m/2}(\mathbb{R}^m)$$

represents, in fact, an interpolation problem. In this way

*) Translated by the editors

**) Cf. also the contribution of Yu.V.Egorov to these proceedings we are able to obtain easily the estimates derived recently in the papers [4-6] - cf. the estimate (26) below. Moreover, the interpolative approach yields estimates optimal with respect to the coupling constant α - cf. the inequality (35). Finally, we demonstrate that the interpolation makes it possible to get an extensive family of estimates containing a functional parameter. These results are useful in the case of strongly anisotropic potentials. On the other hand, it should be mentioned that interpolation usually does not provide optimal constants in the estimates. This report is sketchy; we shall not discuss here a number of related problems where the interpolative approach can be also used.

1. We shall investigate the Schroedinger operator

$$- \Delta - V(x) \qquad (1)$$

in $L_2(\mathbb{R}^m)$, $m \geq 3$, with a potential V decaying at infinity (precise assumptions about V will be formulated below). The number of negative eigenvalues of the operator (1) will be denoted as N(V). We suppose $V(x) \geq 0$, in the opposite case V should be replaced in the estimates of N(V) by its positive part V_+. In addition to (1), we shall discuss the spectrum of the quadratic-form ratio

$$\frac{(Vu,u)}{\int |\nabla u|^2 dx}, \qquad u \in \mathscr{H}^1(\mathbb{R}^m) \qquad (2)$$

$$\mathscr{H}^1(\mathbb{R}^m) = \{ u \in H^1_{loc}(\mathbb{R}^m) :$$

$$\int (|\nabla u|^2 + |x|^{-2}|u|^2)dx < \infty \} \qquad (3)$$

The set $\mathscr{H}^1(\mathbb{R}^m)$, $m \geq 3$, represents a complete Hilbert space with respect to the scalar product

$$\int \nabla u \overline{\nabla v}\, dx$$

The corresponding norm is equivalent to the norm defined by the integral in (3). We remark also that the set $C_0^\infty(\mathbb{R}^m)$ is dense in $\mathscr{H}^1(\mathbb{R}^m)$.

Let λ_k denote the succesive maxima of the ratio (2). In other words, λ_k is an eigenvalue of the operator $T(V)$ associated with the norm

$$(Vu,u) = \int V\ |u|^2 dx$$

in the Hilbert space $\mathscr{H}^1(\mathbb{R}^m)$. The equality

$$N(V) = \operatorname{card}\{\ k;\ \lambda_k > 1\ \} =: n(V)$$

is established in the standard way (cf., e.g., [7],[2]). For any assumptions about the potential V, therefore, estimation of $N(V)$ is reduced to estimation of the quantity $n(V)$.

After we have passed to the spectral problem for the ratio (2), it is convenient to generalize its setting and suppose that V is a complex-valued function. The operator $T(V)$ is then, generally speaking, non-selfadjoint; it is essential that the map

$$\Pi\ :\ \ V \longrightarrow T(V) \qquad (4)$$

is linear.

Let B and S_∞ denote the sets of bounded and compact operators respectively. For $T \in S_\infty$ we set

$$\nu(s;T) = \operatorname{card}\ \{\ k\ :\ s_k(T) > s\ \}\ ;\ s > 0 \qquad (5)$$

where $s_k = s_k(T)$ are successively ordered singular numbers [8] of the operator T. Instead of $\nu(s,T(V))$ we shall write $\nu(s,V)$. If $V(x) > 0$ and the corresponding constant $\alpha > 0$, then

$$N(V) = n(V) = \nu(1;V) \qquad N(\alpha V) = n(\alpha V) = \nu(\alpha^{-1};V) \tag{6}$$

Hence the problem is reduced to estimation of the function $\nu(s;V)$.

2. Below we apply the complex and real interpolation methods. Let us describe the corresponding scales of spaces and the action of interpolation functors.

Lorentz spaces $L_{\sigma,r}(\psi)$. Let ψ be a measurable almost everywhere, finite and positive function on $\mathbb{R}^m$, and let $\rho = \rho_\psi$ be the measure determined by the weight ψ, i.e.,

$$\rho(\delta) = \int_\delta \psi \, dx .$$

With each function f on $\mathbb{R}^m$ which is almost everywhere finite we associate a non-increasing function on $\mathbb{R}_+$:

$$\mu_f(s) = \rho\{ x \in \mathbb{R}^m : |f(x)| > s \} , \ \rho = \rho_\psi$$

By definition, f belongs to $L_{\sigma,r}(\psi)$ for $1 \le r \le \infty$, if the quantity

$$\|f\|^r_{L_{\sigma,r}(\psi)} := \int_0^\infty (s\mu_f^{1/\sigma}(s))^r \, s^{-1} ds , \ 1 \le r < \infty \tag{7}$$

$$\|f\|_{L_{\sigma,\infty}(\psi)} := \sup_{s>0} s \, \mu_f^{1/\sigma}(s) \tag{8}$$