Lecture Notes in Mathematics

Edited by A. Dold and B. Eckmann

966

Algebraic K-Theory

Proceedings, Oberwolfach, 1980
Part I

Edited by R. Keith Dennis

Springer-Verlag
Berlin Heidelberg New York

Lecture Notes in Mathematics

Edited by A. Dold and B. Eckmann

966

Algebraic K-Theory

Proceedings of a Conference
Held at Oberwolfach, June 1980
Part I

Edited by R. Keith Dennis

Springer-Verlag
Berlin Heidelberg New York 1982

Editor

R. Keith Dennis
Mathematics Department, Cornell University
Ithaca, NY 14853, USA

AMS Subject Classifications (1980): 18F25, 12A62, 13D15, 16A54, 20G10

ISBN 3-540-11965-5 Springer-Verlag Berlin Heidelberg New York
ISBN 0-387-11965-5 Springer-Verlag New York Heidelberg Berlin

Printing and binding: Beltz Offsetdruck, Hemsbach/Bergstr.
2146/3140-543210

Introduction*

At one time it was possible to invite everyone interested in algebraic K-theory and its varied applications to one conference. However, that is no longer the case due to the enormous growth of the field. For that reason the algebraic K-theory conference held in June of 1980 at the Forschungsinstitut Oberwolfach was to be primarily concerned with lower algebraic K-theory and some limited aspects of higher K-theory. As can be seen from the List of Talks and the Table of Contents, this restriction was not strictly followed, but it did contribute to the success of the conference by serving as a focal point. The papers appearing in these Proceedings are not so limited in scope and reflect the broad interests of the participants. The contents of the two volumes are roughly divided along the following lines: the first volume consists of papers which are either algebraic K-theory proper or are very closely connected with it (in the view of the editor) while the second volume contains those papers which are either applications of algebraic K-theory to other fields or those whose connections with K-theory are less direct.

Many have contributed to the appearance of this volume and I am deeply grateful for their help. In particular, I owe thanks to Dan Grayson for writing up results of Quillen on finite generation, and to Daniel Quillen for allowing their publication here. I would like to thank Howard Hiller and Ulf Rehmann for preparing their excellent survey talks for publication at my request. Mike Stein was a great help in providing information in regards to organizing a conference and editing its proceedings. Clay Sherman and Wilberd van der Kallen provided many hours of help in ways too numerous to mention. The Mathematics Departments at the Universität Bielefeld, Cornell University, and most of all, Texas Tech University, were of great help in preparing these Proceedings for publication. As usual, the staff at the Forschungsinstitut Oberwolfach kept things running smoothly during the conference. The existence of this conference was assured by one person: Winfried Scharlau. He took the initiative at the crucial time.

R. Keith Dennis

---

* Editors' note: for the sake of completeness we reproduce here the Introduction which appears in Part I of these proceedings (LNM 966) as well as the complete list of talks, and the Contents of both Part I and Part II.

List of Talks

Monday, June 16, 1980

M. Ojanguren, Quadratic forms and K-theory

R. Oliver, SK_1 of p-adic group rings

C. Weibel, Mayer-Vietoris sequences

D. Carter, Word length in $SL_n(\mathcal{O})$

W. van der Kallen, Which $\mathcal{O}$?

Tuesday, June 17, 1980

U. Stuhler, Cohomology of arithmetic groups in the function field case

C. Soulé, Higher p-adic regulators

H. Lindel, The affine case of Quillen's conjecture

T. Vorst, The general linear group of polynomial rings over regular rings

H. Hiller, Affine algebraic K-theory

F. Waldhausen, Informal session on K-theory of spaces

Wednesday, June 18, 1980

A. O. Kuku, A convenient setting for equivariant higher algebraic K-theory

R. W. Sharpe, On the structure of the Steinberg group $St(\Lambda)$

F. Keune, Generalized Steinberg symbols

Thursday, June 19, 1980

K. Kato, Galois cohomology and Milnor's K-groups of complete discrete valuation fields

J. Hurrelbrink, Presentations of $SL_n(\mathcal{O})$ in the real quadratic case

F. Orecchia, The conductor of curves with ordinary singularities and the computation of some K-theory groups

A. Suslin, Stability in algebraic K-theory

J. M. Shapiro, Relations between the Milnor and Quillen K-theory of fields

E. Friedlander, Informal session on etale K-theory

Friday, June 20, 1980

U. Rehmann, The congruence subgroup problem for $SL_n(D)$

A. Bak, The metaplectic and congruence subgroup problems for classical groups G

G. Prasad, The local and global metaplectic conjecture

C. Kassel, Homology of $GL_n(\mathbb{Z})$ with twisted coefficients

J. Huebschmann, Is there a "large" Steinberg group?

W. Pardon, A "Gersten conjecture" for Witt groups and Witt groups of regular local rings

Publisher's note: The seven papers I2, I17, I18, I19, II1, II9, II17 were received by the publisher from the editor of these volumes in July 1982. All other papers had been submitted by June 1981.

TABLE OF CONTENTS

PART I

For the convenience of the reader we list here also the contents of Part II of these proceedings, which appear in Lecture Notes in Mathematics vol. 967.

PART II

ELEMENTS OF SMALL ORDER IN K_2F

Jerzy Browkin

<u>1. Introduction</u>. Let ℓ be a prime number and let ζ_ℓ be a primitive ℓ-th root of unity. J. Tate proved [1] that if a global field F contains ζ_ℓ then every element of order ℓ in K_2F is of the form $\{a,\zeta_\ell\}$ for some $a \in F^*$. It follows that for every positive integer n and every global field F with $\zeta_n \in F$, then every element of order n in K_2F is of the form $\{a,\zeta_n\}$, $a \in F^*$.

In the present paper we investigate elements in K_2F of order n, where $n|12$, even without the assumption that $\zeta_n \in F$.

<u>2. Elements given by cyclotomic polynomials</u>. Let

$$X_n(x) = \prod_{\substack{1 \le k \le n \\ (k,n)=1}} (1 - \zeta_n^k x)$$

be the n-th cyclotomic polynomial.

<u>Lemma 1.</u> For every field F and every positive integer n we have $\{a,X_n(a)\}^n = 1$, provided $a,\ X_n(a) \in F^*$.

<u>Proof</u>. We proceed by induction on n. For $n = 1$ we have $X_1(x) = 1 - x$, and the lemma is obvious.

Now let $n > 1$, and suppose that $\{a,X_d(a)\}^d = 1$ for every $d|n$, $1 \le d < n$ provided $a,\ X_d(x) \in F^*$. If $a \in F^*$ satisfies $a^n \neq 1$, then from

$$1 - x^n = \prod_{d|n} X_d(x)$$

and the inductive assumption it follows that

$$1 = \{a^n, 1 - a^n\} = \prod_{d|n} \{a^d, X_d(a)\}^{n/d} = \{a,X_n(a)\}^n .$$

If $a^n = 1$ and $X_n(a) \neq 0$, then evidently

$$\{a,X_n(a)\}^n = \{1,X_n(a)\} = 1.$$

Theorem 1. For every field $F \neq \mathbf{F}_2$ and $n = 1, 2, 3, 4$ or 6 if $\zeta_n \in F$, then every element $\{a,\zeta_n\}$ in K_2F can be written in the form $\{b,X_n(b)\}$ for some $b \in F^*$ satisfying $X_n(b) \neq 0$.

Proof. For $n = 1$ we have $\{a,\zeta_n\} = 1$ and $\{b,X_n(b)\} = \{b,1-b\} = 1$ for $b \neq 0, 1$.

For $n = 2$ we have

$\{a,\zeta_2\} = \{a,-1\} = \{-1,a\} = \{a-1,a\} = \{a-1,X_2(a-1)\}$.

Thus for $a \neq 1$ it is sufficient to take $b = a-1$, and for $a = 1$ it is sufficient to take $b = 1$.

For $n = 3$, 4 and 6 we have $X_n(x) = (1 - \zeta_n x)(1 - \zeta_n^{-1}x)$. Hence

$$\begin{aligned}\{b,X_n(b)\} &= \{b,1 - \zeta_n b\}\{b,1 - \zeta_n^{-1}b\} \\ &= \{\zeta_n^{-1},1 - \zeta_n b\}\{\zeta_n,1 - \zeta_n^{-1}b\} \\ &= \left\{\zeta_n,\frac{1 - \zeta_n^{-1}b}{1 - \zeta_n b}\right\} .\end{aligned}$$

Thus it is sufficient to determine b from the equation

$$a = \frac{1 - \zeta_n^{-1}b}{1 - \zeta_n b} .$$

We obtain $b = \zeta_n(a-1)(\zeta_n^2 a - 1)^{-1}$. Consequently if $a \neq 1, \zeta_n^{-1}$, then the b given above satisfies the theorem. If $a = 1$ or ζ_n^{-2}, then it is sufficient to take $b = 1$, because $\{\zeta_n^{-2},\zeta_n\} = 1$.

Corollary 1. For every global field F and $n = 1, 2, 3, 4$ or 6 if $\zeta_n \in F$, then every element of order n in K_2F can be written in the form $\{b,X_n(b)\}$, where $b \in F^*$ satisfying $X_n(b) \neq 0$.

Theorem 2. For every field $F \neq \mathbf{F}_2$ and $n = 1, 2, 3, 4$ or 6 the set G_n of elements $\{a,X_n(a)\}$, where $a, X_n(a) \in F^*$, is a subgroup of K_2F.

Proof. Let us observe that $1 \in G_n$. Namely $\{a,X_n(a)\} = 1$, where

$a = 1$, if $n = 2, 4$, char $F \neq 2$ or $n = 6$,

$a = -1$, if $n = 3$,

$a \neq 0, 1,$ if $n = 2, 4,$ $\operatorname{char} F = 2$ or $n = 1$.

Thus G_n is non-empty, and in view of Lemma 1 it is sufficient to prove that the set G_n is closed under multiplication:

If $a, b, X_n(a), X_n(b) \in F^*$, then there exists $c \in F^*$ such that $X_n(c) \neq 0$ and

$$\{a,X_n(a)\}\{b,X_n(b)\} = \{c,X_n(c)\} . \tag{1}$$

For $n = 1$ we have $G_n = 1$. Let $n = 2$. If $a = b$, then the left-hand side of (1) is equal to 1. If $a \neq b$, then $c = \frac{a-b}{1+b}$ satisfies (1) because

$$\{a,1+a\}\{b,1+b\} = \{-1,1+a\}\{-1,1+b\} = \{-1,\tfrac{1+a}{1+b}\} = \{-1,1+c\} = \{c,1+c\} .$$

Now let $n = 3, 4,$ or 6. Put $e_n = 1, 0, -1$ for $n = 3, 4, 6$ respectively. Then $X_n(x) = x^2 + e_n x + 1$. It is easy to verify that

$$X_n(x^{-1}) = x^{-2}X_n(x) \quad \text{and} \quad X_n(-x-e_n) = X_n(x) .$$

Therefore, if $b = a^{-1}$, then

$$\{a,X_n(a)\}\{b,X_n(b)\} = \{a,X_n(a)\}\{a^{-1},a^{-2}X_n(a)\} = \{a^{-1},a^{-2}\} = 1,$$

and if $b = -a-e_n$, then

$$\{a,X_n(a)\}\{b,X_n(b)\} = \{a,X_n(a)\}\{-a-e_n,X_n(a)\} = 1,$$

because $a(-a-e_n) + X_n(a) = 1$.

Now let $b \neq a^{-1}, -a-e_n$. We shall prove that $c = \frac{ab-1}{a+b+e_n}$ satisfies (1). It is easy to check that

$$X_n(a)X_n(b) = (a+b+e_n)^2 X_n(c) . \tag{2}$$

Since

$$\frac{X_n(a)}{a(a+b+e_n)} + \frac{c}{a} = 1 ,$$

then

$$\left\{ \frac{X_n(a)}{a(a+b+e_n)} , \frac{c}{a} \right\} = 1.$$

It follows that

$$\{a,X_n(a)\} = \left\{c, \frac{X_n(a)}{a(a+b+e_n)}\right\} \{a,a(a+b+e_n)\} ,$$

and similarly

$$\{b,X_n(b)\} = \left\{c, \frac{X_n(b)}{b(a+b+e_n)}\right\} \{b,b(a+b+e_n)\} .$$

Multiplying the last two equalities we infer in view of (2) that the left-hand side of (1) is equal to

$$\{c, \frac{1}{ab} X_n(c)\}\{a,a\}\{b,b\}\{ab,a+b+e_n\} =$$
$$= \{c,X_n(c)\}\{ab,c\}\{a,-1\}\{b,-1\}\{ab,a+b+e_n\}$$
$$= \{c,X_n(c)\}\{ab,-c(a+b+e_n)\}$$
$$= \{c,X_n(c)\}\{ab,1-ab\}$$
$$= \{c,X_n(c)\} .$$

3. Elements of small order in K_2Q.

Theorem 3. In K_2Q every element of order 3 has the form $\{a,X_3(a)\}$ for some $a \in Q^*$.

Proof. Let G be the group of elements of the form $\{a,a^2 + a + 1\}$, where $a \in Q^*$. By a theorem of Tate we have an isomorphism given by tame symbols and the Hilbert symbol at the real place:

(3) $$f: K_2Q \longrightarrow Z/2Z \oplus \mathbb{F}_3^* \oplus \mathbb{F}_5^* \oplus \cdots \quad .$$

We proceed by induction. In $Z/2Z$ there are no elements of order 3. Let $u \in K_2Q$ have order 3 and

$$f(u) \in Z/2Z \oplus \mathbb{F}_3^* \oplus \mathbb{F}_5^* \oplus \cdots \oplus \mathbb{F}_p^*$$

for some prime number p. Suppose that the tame symbol ∂_p corresponding to p satisfies $\partial_p(u) \neq 1$. Then $p \equiv 1 \bmod 3$, hence $p = a^2 + ab + b^2$ for some $a, b \in Z$, $0 < |a|,|b| < p$.

Let $w = \{\frac{a}{b},\left(\frac{a}{b}\right)^2 + \frac{a}{b} + 1\} = \{\frac{a}{b},\frac{p}{b^2}\}$. Thus $w \in G$. We have

$$\partial_p(w) = \partial_p\{\frac{a}{b},\frac{p}{b^2}\} = \partial_p\{\frac{a}{b},p\} = \frac{a}{b} \bmod p .$$

Moreover $\frac{a}{b} \neq 1 \bmod p$ and $w^3 = 1$.

In $\mathbb{F}_p^*$ there are exactly two elements of order 3. Consequently

$$\partial_p(u) = \partial_p(w) \quad \text{or} \quad \partial_p(u) = \partial_p(w^2).$$

Thus by the inductive assumption we have $uw^{-1} \in G$ or $uw^{-2} \in G$. Hence $u \in G$.

Theorem 4. In K_2Q every element of order 4 has the form $\{a,a^2+1\}v$, where $a \in Q^*$, and $v \in K_2Q$, $v^2 = 1$.

Proof. Let G be the group of elements of K_2Q of the form given in the theorem. We use the homomorphism f given by (3) and proceed by induction. In $Z/2Z$ there is no element of order 4. Let $u \in K_2Q$ have order 4 and let

$$f(u) \in Z/2Z \oplus F_3^* \oplus \cdots \oplus F_p^*$$

for some prime number p. Suppose that $\partial_p(u) \neq 1$. Then multiplying u by some element of order 2 we may assume that $\partial_p(u)$ has order 4. Consequently $p \equiv 1 \bmod 4$, and $p = a^2 + b^2$ for $a, b \in Z$, $0 < a, b < p$.

Let $w = \{\frac{a}{b}, \left(\frac{a}{b}\right)^2 + 1\} = \{\frac{a}{b}, \frac{p}{b^2}\}$. Evidently $w \in G$ and $\partial_p(w) = \partial_p\{\frac{a}{b}, p\} = \frac{a}{b} \bmod p$. We have $\frac{a}{b} \neq \pm 1 \bmod p$, and consequently $\partial_p(w)$ has order 4. Thus $\partial_p(u) = \partial_p(w)$ or $\partial_p(u) = \partial_p(w^{-1})$. By the inductive assumption we obtain that $uw^{-1} \in G$ or $uw \in G$, and consequently $u \in G$.

Corollary 2. In K_2Q every element of order 6 or 12 has the form

$$\{a, a^2 + a + 1\}\{b, b+1\} \quad \text{or} \quad \{a, a^2+1\}\{b, b^2+b+1\}\{c, c+1\}$$

respectively.

Proof. Every element of order 6 (respectively 12) is a product of elements of orders 2 and 3 (respectively 3 and 4).

Remarks. One can conjecture that

(1) Theorems 1 and 2 do not hold for any other values of n (and all fields F).

I do not know the answer for example when $n = 5$ and $F = Q$.

(2) Theorems 3 and 4 hold for all fields.

J. Urbanowicz [2] generalized Theorem 3 to all number fields.

References.

[1] J. Tate, Relations between K_2 and Galois cohomology, Inventiones Math. 36(1976), 257-274.

[2] J. Urbanowicz, Thesis, Warsaw University, 1980.

Institute of Mathematics
Warsaw University
PL-00-901 Warsaw
Poland

HOCHSCHILD HOMOLOGY AND THE SECOND OBSTRUCTION FOR PSEUDOISOTOPY

R. Keith Dennis[1]

Texas Tech University, Lubbock, Texas

and

Kiyoshi Igusa[2]

Brandeis University, Waltham, Mass.

INTRODUCTION

In this paper we investigate the relative stable K-theory group $K_2(R, A)$ which is defined for a ring R and an R-bimodule A. This group is related to K_2 of a square zero ideal, the space of pseudoisotopies of a smooth manifold, and to Waldhausen's algebraic K-theory of a space.

The main result of this paper is the computation $K_2(R, A) \cong H_1(R, A)$ where the latter is the (easily computable) Hochschild homology of R with coefficients in A. We start with A. Hatcher's definition of $K_2(R, A)$ in terms of generators and relations and by proving a sequence of isomorphisms we reach $H_1(R, A)$. In parts A and B we present two such proofs with different intermediary groups. This has resulted in the long list of isomorphic groups given in the appendix.

In part C we present the relationship between $K_2(R, A)$ and the second obstruction for π_1 of the space of pseudoisotopies of a smooth manifold. In part D we present briefly the relationship with the work of F. Waldhausen and C. Kassel.

ACKNOWLEDGEMENTS

Many of the ideas presented here are originally due to A. Hatcher. It was Hatcher who gave the original definition of $St(R, A)$ and $K_2(R, A)$ which arises naturally in the study of pseudoisotopies of smooth manifolds. Hatcher explained this re-

1. Research supported by NSF grant no. MCS78-00987.

2. Research supported by NSF grant no. MCS79-02939.

lationship to the second suthor who was then his student in Spring 1975 [HO]. Later that year in a letter [H1] Hatcher constructed a map from $H_1(R, A)$ to $K_2'(R, A)$ and conjectured that this was an isomorphism. In another letter [H2] he gave the definition of $K_i^n(R, A)$ (D.1.1) and conjectured that it was independent of n.

The proof given in part B was written between the two letters from Hatcher. In this proof essential use was made of the symbol $< , >$ of [S-D] and the curious map δ (B.3.1) the idea for which originates from a letter from L.G. Brown [B].

The proof given in part A was written in July 1977 with the help of F. Waldhausen who supplied the idea for the proof of A.3.4. It should be noted that the Morita invariance of Hochschild homology is evident from its functorial description. (See, e.g., [M] Chap. IX.) Section 1 of part A is based on an argument given in [Mi] pp. 41-51. The idea of universal G-central extension is also a special case of the universal central relative extension of [L].

Some of the results in part C have been greatly generalized by T. Goodwillie who has devised new methods [G] to compute the homotopy groups of the fiber of the map $P(X) \to P(BG)$ where $G = \pi_1 X$.

C. Kassel has also done some work related to this paper. In his thesis [K4] he gives a short presentation of almost all the material in parts A and B giving us credit of course. However, we should acknowledge that he discovered much of this independently.

Finally, we should note that part A has previously appeared as a preprint entitled "A proof of a theorem of R. K. Dennis." Needless to say the proofs in parts A and B have been recently revised.