Control Dynamics of ROBOTIC MANIPULATORS

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1986

ACADEMIC PRESS, INC.

Harcourt Brace Jovanovich, Publishers

Orlando San Diego New York Austin
Boston London Sydney Tokyo Toronto

ACADEMIC PRESS, INC.
Orlando, Florida 32887

United Kingdom Edition published by
ACADEMIC PRESS INC. (LONDON) LTD.
24–28 Oval Road, London NW1 7DX

Library of Congress Cataloging in Publication Data

Skowroński, Janisław M.
Control dynamics of robotic manipulators.

Bibliography: p.
Includes index.
1. Robotics. 2. Manipulators (Mechanism) I. Title.
TJ211.S59 1986 629.8'92 86-7996
ISBN 0–12–648130–X (alk. paper)

PRINTED IN THE UNITED STATES OF AMERICA

86 87 88 89 9 8 7 6 5 4 3 2 1

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Preface

In a large number of well-known universities, the mechanical sciences (applied mechanics, mechanical systems dynamics and control, aero and hydro mechanics, elasticity, etc.) have separated from mechanical engineering departments into independent teaching units or have joined the systems sciences. There are serious arguments for doing so, but there is also definitely at least one substantial negative in such an action, and that is the resulting lack of cooperation between technological-production oriented engineering and its fundamental counterpart, mechanical systems dynamics. The technologists miss out on a broader scope of options and thus on flexibility in design, and the theoreticians lose reference to real problems. The gap widens as it spreads from research to teaching and then to textbooks, and this may well impede future development.

Robotics, quite naturally developed so far by the production oriented groups, begins to suffer from the above fate, particularly in view of its growing need to be applicable almost everywhere. Not only must manipulators be automatically controlled to reach from A to B, but they must also be made to follow a stipulated smooth path, avoiding various stationary or moving obstacles. In addition they are required to do this quite accurately in a specified time and space and at high speed, in cooperation or competition with other machines or other manipulators for that matter. Often they also need to do it optimally with respect to perhaps several cost functionals.

These tasks should be achieved while working in a variable environment and subject to various, frequently uncertain, dynamic payloads and hence also to unknown dynamical structural forces. Manipulators must thus be stabilized and made robust against unpredictable conditions, as well as capable of attaining the objectives mentioned. This yields a need for feedback and adaptive controllers, as well as perhaps a self-organizing structure.

Such manipulators become a complex, strongly nonlinear and strongly coupled system with many degrees of freedom. Control based on a linearized or otherwise simplified model loses effectiveness to the degree that the real working procedure deviates from the conditions assumed for simplification. This can lead to displacements of a manipulator from the desired trajectory in spite of all control efforts. It can even make the motion of the manipulator incompatible with such a trajectory, for instance, if the equilibria of the real nonlinear system do not coincide with the single equilibrium of the linearized model.

All of the above requires delving more deeply into rather sophisticated basic research in control and systems dynamics, not only to use it but also to extend it or at least to adjust it for our purposes. It means the situation in development of manipulators has matured to fundamental studies, and there is a need to investigate which part of the wide range of control dynamical results can be applied.

There are several excellent textbooks on robotic manipulators which attempt and largely succeed in presenting the material in a unified way with some of the above in mind—for example, Paul [1], Coiffet [1], and Snyder [1]; however, they stop on "unfinished" problems currently under research or requiring research. That is exactly where we would like to start, indicating to the postgraduate student or design engineer which branches and topics in control and systems dynamics are applicable and perhaps suggesting appropriate methods. Obviously no book can pretend to do the above regarding all the topics involved either in manipulator theory or in its control system mechanical background. The selection must be a matter of what seems to be more urgently needed (judging by its research popularity) and obviously biased by the experience and preferences of the author and his circle of collaborators.

We refer briefly to various existing and possible models (Chapter 1) and energy relations needed later (Chapter 2), and introduce elementary nonlinear controllability and stabilization conditions (first part of Chapter 3). Then we discuss control under uncertainity, both "worst-case design" and adaptive, referring first to stabilization (Chapter 3), then to various elementary objectives like reaching, real-time reaching, maneuvering, capture (handling) and optimal capture of an object (target), planned path tracking, and model reference adaptive control (Chapter 4). Avoidance of stationary and moving obstacles is the next main topic (Chapter 5). We close with adaptive identification of states and parameters (Chapter 6).

Acknowledgments

The author is indebted to Professors M. Corless, H. Flasher, W. J. Grantham, G. Leitmann, W. E. Schmitendorf, R. J. Stonier, and T. L. Vincent for cooperation leading the the results included in this book, as well as to some of the above-mentioned colleagues for critical comments on the text.

The author is also indebted to Professor T. Maxworthy and the Mechanical Engineering Department at the University of Southern California for encouragement and help in teaching the course.

Contents

Chapter 1

Mechanical Models

1.1 MODULAR RP-UNIT MANIPULATOR

The robot manipulators may be presently modeled as complex (branching) open or closed chain machines, but for the sake of basic dynamic studies they still are best considered consisting of simple open kinetic chains of n material links, each link with a single degree of freedom (DOF), connected together by joints. Such a manipulator may have more than n DOF, the additional due to orientation of the base and the gripper, but the DOF of the links are fundamental to our study. The joints are either *revolute* (rotational) coded R, or *prismatic* (translational) coded P. So it seems that a modular two-link RP-unit manipulator with two DOF, one rotation and one translation, can serve a twofold purpose: as a unit for composing up a suitable machine, and as an instructive example for studying both dynamics and control in the general case. It also seems that some fundamental notions of our study are best introduced and explained first on such an example.

Figure 1.1 displays the set-up. The link 1 is posed at the *base* with underlying revolute joint 1, the link 2 is connected to the first by a prismatic joint 2. Obviously the order of joints could have been reversed without influencing the generality. The mass of each link and of the corresponding joint is lumped into the mass point m_i, $i = 1, 2$, located at the end of a link, the first mass at the fixed distance r_1 from the joint 1, the second at the variable distance r. The mass m_1 rotates about the joint 1 and the mass m_2 translates together with the link 2 respective to the link 1, the motion allowed by the joint 2. These DOF are measured in terms of either *Cartesian* or *Langrangian coordinates*, the values of which give time instantaneously the *configuration* of the manipulator arm.

We place the Cartesian reference frame $0\xi_0\eta_0\zeta_0$ of the "world" coordinates fixed to the base, as shown in Fig. 1.1, with the origin at joint 1. It will thus also be called the *inertial* or, more frequently, *base system*. We also embed a Cartesian reference frame $0_i\xi_i\eta_i\zeta_i$ in each link $i = 1, 2$. These are body coordinates of the link concerned, called briefly *link coordinates*. We let the origin 0_i be at the joint $i + 1$, while the ζ_i axis is parallel to the axis of joint i

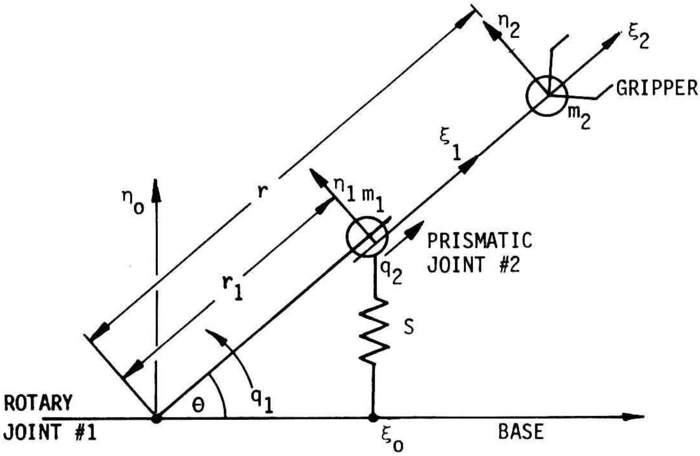

Fig. 1.1

(axis of rotation or translation). This means that 0_1 is placed at the end of link 1, that is, at joint 2.

The motion of a link is considered in terms of the motion of its link coordinates $0_i\xi_i\eta_i\zeta_i$ with respect to some reference frame. In general such motion is determined by three changes of *position* along coordinates of this frame (translation of 0_i) and three changes of *orientation* (rotation of $0_i, \xi_i, \eta_i, \zeta_i$), as well as the corresponding velocities. Obviously, the motion of link 1 is referenced to the base frame $0\xi_0\eta_0\zeta_0$. In our case link 1 rotates only about the ζ_0 axis with the position and velocities of m_1 $(\xi_{01}, \eta_{01}, \zeta_{01})$ specified by

$$\begin{aligned} \xi_{01}(t) &= r_1 \cos\theta(t), & \dot{\xi}_{01}(t) &= -r_1 \sin\theta(t)\cdot\dot{\theta}(t), \\ \eta_{01}(t) &= r_1 \sin\theta(t), & \dot{\eta}_{01}(t) &= r_1 \cos\theta(t)\cdot\dot{\theta}(t), \\ \zeta_{01}(t) &\equiv 0, & \dot{\zeta}_{01}(t) &\equiv 0, \end{aligned} \tag{1.1.1}$$

where t is time, with $t \geq t_0$, which is an initial (reference) instant, $t_0 \in R$.

The motion of any other link $i > 1$ may be investigated by referencing it *relative* to the frame fixed at the previous link $0_{i-1}\xi_{i-1}\eta_{i-1}\zeta_{i-1}$. Such a relative reference applied sequentially leads to indirect reference to the base coordinates $0\xi_0\eta_0\zeta_0$. Alternatively, we may reference the motion of each link i *directly* to $0\xi_0\eta_0\zeta_0$.

In the case of our example, the translation of m_2 $(\xi_{12}, \eta_{12}, \zeta_{12})$ presently referred to $0_0\xi_0\eta_0\zeta_0$ gives

$$\begin{aligned} \xi_{02}(t) &= r(t)\cos\theta(t), \\ \eta_{02}(t) &= r(t)\sin\theta(t), \\ \zeta_{02}(t) &\equiv 0; \end{aligned} \tag{1.1.2}$$

and the velocities

$$\begin{aligned}\dot{\xi}_{02}(t) &= \dot{r}(t)\cos\theta(t) - \dot{\theta}(t)r(t)\sin\theta(t),\\ \dot{\eta}_{02}(t) &= \dot{r}(t)\sin\theta(t) + \dot{\theta}(t)r(t)\cos\theta(t),\\ \dot{\zeta}_{02}(t) &\equiv 0,\end{aligned} \tag{1.1.3}$$

as the direct-to-base reference approach.

Note that (1.1.3) may be written as

$$\begin{bmatrix}\dot{\xi}_{02}\\ \dot{\eta}_{02}\end{bmatrix} = \begin{bmatrix}\cos\theta & -r\sin\theta\\ \sin\theta & r\cos\theta\end{bmatrix}\begin{bmatrix}\dot{r}\\ \dot{\theta}\end{bmatrix}, \tag{1.1.4}$$

where

$$J(r,\theta) = \begin{bmatrix}\cos\theta & -r\sin\theta\\ \sin\theta & r\sin\theta\end{bmatrix} \tag{1.1.5}$$

is the Jacobian of the transformation between the variables

$$(r,\theta) \to (\xi_{02}, \eta_{02}, \zeta_{02}).$$

More generally, the standard routine in manipulator kinematics (see Paul [1]) is to use a transformation matrix A_i, $i = 1, 2$, describing the relative translation and/or rotation between $0_i\xi_i\eta_i\zeta_i$ and $0_{i-1}\xi_{i-1}\eta_{i-1}\zeta_{i-1}$. Then the position and orientation of the link i in base coordinates are given by the matrix product $T_i = A_1 \cdot A_2 \cdot \cdots \cdot A_i$. The matrix A_i transforms the position vector $\bar{r}_{i-1}$ of the mass point m_{i-1} into $\bar{r}_i$ of $m_i: \bar{r}_i = A_i\bar{r}_{i-1}$. The matrix may include a rotation matrix (direction cosines) and/or translation matrix (components of translations vector). Depending upon the kinematic method employed, these vectors and matrices may be *either* three dimensional and 3×3, immediately generalizing (1.1.1), (1.1.2), *or* four dimensional called *homogeneous* with $\bar{r}_i = (a_i, b_i, c_i)$ represented by the matrix $[\xi_i\eta_i\zeta_i d_i]^{\mathrm{T}}$, $a_i = \xi_i/d_i$, $b_i = \eta_i/d_i$, $c_i = \zeta_i/d_i$ and the transformation matrices 4×4.

In our example A_1 is a rotation matrix only, while A_2 represents only translations

$$A_1 = \begin{bmatrix}\cos\theta & -\sin\theta & 0 & 0\\ \sin\theta & \cos\theta & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{bmatrix}, \tag{1.1.6}$$

$$A_2 = \begin{bmatrix}1 & 0 & 0 & r\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{bmatrix}. \tag{1.1.7}$$

Then the relation to the base coordinates is given by

$$T_2 = A_1 \cdot A_2 = \begin{bmatrix} \cos\theta & -\sin\theta & 0 & r\cos\theta \\ \sin\theta & \cos\theta & 0 & r\sin\theta \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{1.1.8}$$

Note here that the general homogeneous transformation illustrates the case of rotation first and then translation (see the fourth column of the T_2 matrix). We took the link coordinates positioned similarly to $0\xi_0\eta_0\zeta_0$ for illustrative purposes. A more frequently used convention is to take ζ_i along the axis of rotation or translation. Readers interested in kinematics of manipulators may look for details in Paul [1], Coiffet [1], Snyder [1], or Kooleshov and Lakota [1].

The relative reference approach is also convenient for the fast numerical simulation (see Hollerbach [1]) and thus for speeding up the on-line calculations during the work of the machine, as well as for deriving the kinetostatic relations needed for computer aided modeling (see Stepanenko and Vukobratovic [1]). In both these applications recursive kinematics is used, i.e., calculations begin with a given path of the gripper. We shall return to this topic in Sections 3.1, 4.6, and 5.5.

On the other hand the direct approach relates the *mechanical network* (see Skowronski [2]) of links and appendages (gripper, vehicle) to other objects in the world space $0\xi_0\eta_0\zeta_0$ on which the manipulator works, later called *targets*, or which it must avoid, termed *obstacles*. Once the links and appendages are referred to $0\xi_0\eta_0\zeta_0$, we consider their relation to the objects in this space or, if convenient, simplify matters by taking the space of relative distances between the elements of the network and the objects. Then the origin (zero distance) in that space is discussed as either an attracting target or repelling obstacle, see Sections 4.6 and 5.5.

Moreover representing the robot and its environment as a joint system in $0\xi_0\eta_0\zeta_0$ allows the use of the wealth of methods available in system dynamics, providing an insight into the dynamic behavior of the robot, which is much needed for synthesis, design, and optimization. Such an insight is very poor or totally missing in the numerically based kinetostatic investigation (for a review see Silver [1]).

In this section we simplify the network to the two links concerned, with the last mass m_2 also representing the gripper, and m_1 covering the vehicle as well, provided it is fixed at 0. These approximations fully suit our purpose of control-dynamical studies, and have been justified within the so-called "augmented body" approach (see Liegeois, Khalil, Dumas, and Renaud [1] and Hooker and Margulies [1]).

We turn now to the environmental object (target, obstacle) which may be represented by points in $0\xi_0\eta_0\zeta_0$. In the case of a lumped mass representation, it will be a single mass point; in the case of a larger body, it will be several geometric points specifying the boundary. Let us take a mass point m_3 determined by $\xi_{03}, \eta_{03}, \zeta_{03}$ as the target to be reached by the gripper, and a body determined by the band (*safety zone*)

$$\xi_{04} \leq \xi_0 \leq \xi_{05}, \qquad \eta_{04} \leq \eta_0 \leq \eta_{05}, \qquad \zeta_0 \neq 0 \tag{1.1.9}$$

enveloping an *antitarget* as a set to be avoided.

The objects are either *at rest* or *moving*. We assume here the first case, leaving the second to more general study in Section 1.2. We have $\xi_{03} = a$, $\eta_{03} = b$, $\zeta_{03} = 0$, $\xi_{04} = a_4$, $\xi_{05} = a_5$, $\eta_{04} = b_4$, $\eta_{05} = b_5$, all constants.

Our gripper m_2 will now pursue m_3 $(a, b, 0)$ while m_1, m_2 avoid the band $a_4 \leq \xi_{0i} \leq a_5$, $b_4 \leq \eta_{0i} \leq b_5$, $i = 1, 2$, with the kinematics described by (1.1.1), (1.1.2), and (1.1.4). Observe however that the motion is constrained by the fact that each joint is allowed only one DOF. Indeed, from (1.1.1), (1.1.2), and (1.1.4), we see that the motion depends only on two independent variables $\theta(t)$, $r(t)$, and their derivatives $\dot{\theta}(t) = \omega(t)$ and $\dot{r}(t)$ varying with time $t \geq t_0$. We thus choose them as the Lagrangian *joint coordinates* $q_1 = \theta(t)$, $q_2 = r(t)$ with $\dot{q}_1 = \omega(t)$, $\dot{q}_2 = \dot{r}(t)$. Note that the relative referenced choice $q_2 = r(t) - r_1$, $r_1 = \text{const}$, does not change our case since all derivatives of r_1 vanish. It is easily seen that the motion is completely determined by $q_1(t), q_2(t), \dot{q}_1(t), \dot{q}_2(t)$, or by two vectors: the configuration vector $\bar{q}(t) = (q_1(t), q_2(t))^{\mathrm{T}}$ in the *configuration space* R^2 and the velocity vector $\dot{\bar{q}}(t) = (\dot{q}_1(t), \dot{q}_2(t))^{\mathrm{T}}$ in the corresponding tangent space of generalized velocities.

We thus arrive at the representation of motion in the *phase space* R^{2n}, $n = 2$, by the vector $\bar{x}(t) = (q_1(t), q_2(t), \dot{q}_1(t), \dot{q}_2(t))^{\mathrm{T}}$, $t \geq t_0$, which will later be called the *state vector* of the system, while R^{2n} becomes the *state space* R^N, $N = 2n$.

The reaching ability of the joints, in particular of the gripper m_2, in both configuration and phase spaces is limited. The gripper and the joint 1 can maneuver in a bounded work space in $0\xi_0\eta_0\zeta_0$ restricted by the physical limitations of its design, while the velocities and accelerations are bounded by the bounds on the controlling forces and torques of actuators. The restrictions by design will be most frequently given directly in terms of θ, r in the configuration space. They produce a bounded set Δ_q called the *work region.* Together with the bounded set $\Delta_{\dot{q}}$ in the tangent space restricting the velocities, these two sets generate the *state work region* $\Delta = \Delta_q \times \Delta_{\dot{q}} \subset R^N$, expressed in terms of specified stop limits on $\bar{q}(t)$, $\dot{\bar{q}}(t)$. In our case Δ_q will be bounded only by the design limitation on $q_2 = r$, $q_1 = \theta$ being arbitrary.

Recall that in $0\xi_0\eta_0\zeta_0$ the gripper attempted to reach the target m_3 $(a, b, 0)$ while both joints had been avoiding the obstacles. Because of the obstacles, in order to do a job on a workpiece, say m_3, it becomes important for the arm to achieve such a *configuration of all the joints* (in our case two) such as to make the reaching by the gripper possible. Since a vector $\bar{q}(t)$ in the configuration space represents the positions of all the joints, *we may reach the above-mentioned target configuration by conveniently reaching a desired stipulated point* $\bar{q}$ in R^2, which thus may be called a *configuration target*. It may be easily seen that it is different from m_3 in $0\xi_0\eta_0\zeta_0$ in that it refers to all the joint variables involved in the system, not to the gripper only. With this in mind, we still briefly call it a *target*.

Now the Cartesian target m_3 $(a, b, 0)$ and the Cartesian avoidance band (1.1.6) must be transferred into the joint coordinates, obviously by inverting (1.1.1), (1.1.2), and (1.1.4). Unfortunately, this is not a unique procedure even in our very simple modular case and is obviously less so in general. Indeed the inverses of (1.1.1), (1.1.2) give

$$\begin{aligned} q_2 &= r(t) = \{[\xi_{02}(t)]^2 + [\eta_{02}(t)]^2\}^{1/2}, \\ q_1 &= \theta(t) = \arctan(\eta_{02}(t)/\xi_{02}(t)) = \arctan(\eta_{01}(t)/\xi_{01}(t)) \end{aligned} \tag{1.1.10}$$

which produces a sequence of values for $\theta(t)$. The above gives the configuration target $\mathfrak{T}_q$:

$$q_1 = \arctan(b/a), \qquad q_2 = (a^2 + b^2)^{1/2}, \tag{1.1.11}$$

and the avoidance band as $\mathscr{A}_q$:

$$\begin{aligned} \arctan(b_4/a_4) \le q_1 \le \arctan(b_5/a_5), \\ (a_4^2 + b_4^2)^{1/2} \le q_2 \le (a_5^2 + b_5^2)^{1/2}. \end{aligned} \tag{1.1.12}$$

In order to invert (1.1.4) we differentiate (1.1.10) obtaining

$$2r\dot{r} = 2\xi_{02}\dot{\xi}_{02} + 2\eta_{02}\dot{\eta}_{02}, \qquad \dot{\theta}\sec^2\theta = \frac{-\eta_{02}}{\xi_{02}}\dot{\xi}_{02} + \frac{1}{\xi_{02}}\dot{\eta}_{02}.$$

The latter yields

$$\dot{\theta} = \frac{-\eta_{02}}{\xi_{02}(r^2/\xi_{02}^2)}\dot{\xi}_{02} + \frac{1}{\xi_{02}(r^2/\xi_{02}^2)}\dot{\eta}_{02}.$$

Hence the inverse of (1.1.4) is

$$\begin{aligned} \dot{q}_2 &= \dot{r}(t) = \frac{\xi_{02}(t)}{r(t)}\dot{\xi}_{02}(t) + \frac{\eta_{02}(t)}{r(t)}\dot{\eta}_{02}(t), \\ \dot{q}_1 &= \dot{\theta}(t) = \frac{-\eta_{02}(t)}{r^2(t)}\dot{\xi}_{02}(t) + \frac{\xi_{02}(t)}{r^2(t)}\dot{\eta}_{02}, \end{aligned} \tag{1.1.13}$$

or

$$\begin{pmatrix} \dot{r} \\ \dot{\theta} \end{pmatrix} = \begin{pmatrix} \xi_{02}/r & \eta_{02}/r \\ -\eta_{02}/r^2 & \xi_{02}/r^2 \end{pmatrix} \begin{pmatrix} \dot{\xi}_{02} \\ \dot{\eta}_{02} \end{pmatrix},$$

with the inverted Jacobian matrix

$$J^{-1}(r,\theta)=\begin{pmatrix}\xi_{02}/r & \eta_{02}/r\\ -\eta_{02}/r^2 & \xi_{02}/r^2\end{pmatrix}, \tag{1.1.14}$$

which is a key feature of the "inverse" kinematics and dynamics described later in Chapter 3.

The relation (1.1.13) now serves to establish the corresponding target and the possible avoidance band in the space of generalized velocities to complement (1.1.11) and (1.1.12) to a *target* $\mathfrak{T}$ and an *avoidance set* $\mathscr{A}$ in the phase space. Indeed, if there were a task of attaining some velocities $\dot{\xi}_{02}(t)=c$, and $\dot{\eta}_{02}(t)=d$ by the gripper while at $\mathfrak{T}_q$, in particular to stop there ($c=0$, $d=0$), by (1.1.13) we would aim at $\dot{q}_1=0$, $\dot{q}_2=0$, with the target

$$\mathfrak{T}: q_1=\arctan(b/a), \qquad q_2=(a^2+b^2)^{1/2}, \qquad \dot{q}_1=0, \quad \dot{q}_2=0. \tag{1.1.15}$$

Moreover, if one wished to avoid excessive speeds $\dot{\xi}_{02}=c_4, \dot{\eta}_{02}=d_4, i=1,2$ close to the obstacles, then by (1.1.13) one would have to avoid $\mathscr{A}$ specified by (1.1.12) and

$$\begin{aligned}\dot{q}_1 &\geq \frac{a_4c_4}{(a_4^2+b_4^2)^{1/2}}+\frac{b_4d_4}{(a_4^2+b_4^2)^{1/2}},\\ \dot{q}_2 &\geq \frac{-b_4c_4}{a_4^2+b_4^2}+\frac{a_4d_4}{a_4^2+b_4^2},\end{aligned} \tag{1.1.16}$$

when approach $\mathscr{A}_q$ from below, or (1.1.16) with a_4, b_4 replaced by a_5, b_5 when approaching $\mathscr{A}_q$ from above.

It is now of interest how $\mathfrak{T}$ and $\mathscr{A}$ relate to the work region Δ, which is convenient if made simply connected. The target $\mathfrak{T}$ (1.1.15) fits for a suitable length r of the arm, that is, when $\sqrt{a^2+b^2}<r_{\max}$; $r_{\max}$ provided by the designer. The obstacle along q_2 may obviously be excluded from Δ_q by suitable adjustment of $r_{\min}$, $r_{\max}$, without spoiling the connectedness. However, the angular obstacle on q_1 (1.1.12) has to be covered by Δ_q and made avoidable by means of the controlling torque rather than by exclusion. The velocity obstacles (1.1.16) can be excluded by adjusting the boundaries of $\Delta_{\dot{q}}$ in a similar way.

The motion of our manipulator is generated by two controlling actuators or *drives*, which may be electrical, pneumatic, hydraulic or mixed, depending on the job to be done. These actuators produce *torques* or *forces*, while acting upon the joints, rotary or prismatic, respectively, through some *transmission* that includes clutches, brakes, etc. In our case we let the control variables $u_1(t)$, $u_2(t)$ represent the input from the two actuators producing a torque $Q_1^F(\bar{q},\dot{\bar{q}},u_1)$ on joint 1 and a force $Q_2^F(\bar{q},\dot{\bar{q}},u_2)$ on joint 2, the functions $Q_1^F(\cdot)$, $Q_2^F(\cdot)$ being called *input transmission forces*.

The actuator's job is twofold; it must be *passive* to offset the gravity of links and joints (their own weight as well as the payload) and *active* to perform a specified assignment for the manipulator. It is often convenient to replace, at least in part, the passive job by the static effort of spring forces acting on or between the lumped masses. In our case we offset the gravity $9.81m_1$ and $9.81(m_1 + m_2)$ by setting up the spring force $S = a_0 + aq_1 + bq_1^3$ (see Fig. 1.1) with the free-play $a_0 = -9.81m_1r_1$ and a hardening effect afterward. Elementary mechanics gives the potential energy as

$$\mathscr{V}(\bar{q}) = 9.81m_1r_1 \sin q_1 - 9.81m_1r_1q_1 + \tfrac{1}{2}aq_1^2 + \tfrac{1}{4}bq_1^4 + 9.81m_2q_2 \sin q_1, \tag{1.1.17}$$

and the kinetic energy as

$$T(\bar{q}, \dot{\bar{q}}) = \tfrac{1}{2}m_1r_1^2\dot{q}_1^2 + \tfrac{1}{2}m_2(\dot{q}_2^2 + q_2^2\dot{q}_1^2). \tag{1.1.18}$$

We immediately calculate the Lagrangian $L(\bar{q}, \dot{\bar{q}}) = T(\bar{q}, \dot{\bar{q}}) - \mathscr{V}(\bar{q})$. Then we assume the damping force in the joints to be $\lambda_1|\dot{q}_1|\dot{q}_1$ and $\lambda_2\dot{q}_2$, respectively, with $\lambda_i = \text{const} > 0$, $i = 1, 2$ being the damping coefficients. With the input transmission generally described by $Q_i^F(\bar{q}, \dot{\bar{q}}, u_i)$, $i = 1, 2$, the Lagrange equations of motion (see Section 1.3) immediately give

$$\begin{aligned} &(m_1r_1^2 + m_2q_2^2)\ddot{q}_1 + 2m_2q_2\dot{q}_1\dot{q}_2 + \lambda_1|\dot{q}_1|\dot{q} + 9.81(m_1r_1 + m_2q_2)\cos q_1 \\ &\qquad - 9.81m_1r_1 + aq_1 + bq_1^3 = Q_1^F, \\ &m_2\ddot{q}_2 - m_2q_2\dot{q}_1^2 + \lambda_2\dot{q}_2 + 9.81m_2 \sin q_1 = Q_2^F. \end{aligned} \tag{1.1.19}$$

Now since $(m_1r_1^2 + m_2q_2^2) > 0$, $m_2 > 0$, we may let

$$\begin{aligned} G_1(\bar{q}) &= \frac{9.81(m_1r_1 + m_2q_2)}{m_1r_1^2 + m_2q_2^2}\cos q_1, \\ G_2(\bar{q}) &= 9.81 \sin q_1 \end{aligned} \tag{1.1.20}$$

be *gravity characteristics*, while

$$\begin{aligned} \Psi_1(\bar{q}) &= \frac{-9.81m_1r_1 + aq_1 + bq_1^3}{m_1r_1^2 + m_2q_2^2}, \\ \Psi_2(\bar{q}) &\equiv 0 \end{aligned} \tag{1.1.21}$$

are *elastic* (*spring*) *characteristics* of the system.

Then the sum

$$\Pi_i(\bar{q}) = \Psi_i(\bar{q}) + G_i(\bar{q}), \qquad i = 1, 2, \tag{1.1.22}$$

will be called the *characteristics of potential forces*. In turn, the characteristics of nonpotential forces (centrifugal, Coriolis, damping) will in our case reduce