FOURIER SERIES AND BOUNDARY VALUE PROBLEMS

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PREFACE

This is an introductory treatment of Fourier series and their applications to boundary value problems in partial differential equations of engineering and physics. It is designed for students who have completed the equivalent of one semester of advanced calculus. The physical applications, explained in some detail, are kept on a fairly elementary level.

The first objective is to introduce the concept of orthogonal sets of functions and representations of arbitrary functions in series of the functions of such sets. The most prominent special cases, the representation of functions by trigonometric Fourier series, are given special attention. Fourier integral representations and expansions in series of Bessel functions and Legendre polynomials are also treated.

The second objective is a clear presentation of the classical method of solving boundary value problems with the aid of those representations in series of orthogonal functions. Some attention is given to the verification of solutions and to uniqueness of solutions, for the method cannot be presented properly without such considerations. Other methods are treated in the author's books "Operational Mathematics" and "Complex Variables and Applications."

This edition is an extensive revision of the original 1941 edition of the book. The exposition has been revised throughout. Some additional material has been introduced on differential equations and boundary conditions, uniform convergence, complex-valued functions, Fourier integrals, convergence of Legendre's series, uniqueness of solutions, and other topics. Some rearrangement of topics was found desirable; for instance, partial differential equations of physics are now treated in the first chapter in order to simplify the introduction of other topics.

Additional attention is given to the mathematical analysis, Examples, problems, figures, and bibliography have been revised.

The chapters on Bessel functions and Legendre polynomials, Chapters 8 and 9, are independent of each other. They can be taken up in either order. Chapter 10, on uniqueness of solutions, and Chapter 5, on further properties of Fourier series, as well as some sections of other chapters, can be omitted in order to shorten the course.

In the development of the book through this edition the author acknowledges the helpful comments and encouragement of many teachers and students. Among his local colleagues, Professors R. C. F. Bartels, C. L. Dolph, G. E. Hay, and E. D. Rainville deserve special thanks.

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CONTENTS

CHAPTER 1

PARTIAL DIFFERENTIAL EQUATIONS OF PHYSICS

1. Two Related Problems. We shall be concerned here with two general types of problems. One type deals with the representation of arbitrarily given functions by infinite series of functions of a prescribed set. The other consists of boundary value problems in partial differential equations, with emphasis on equations that are prominent in physics and engineering.

Representations by series are encountered in methods of solving boundary value problems. The theories of those representations can be presented independently. They have such attractive features as the extension of concepts of geometry, vector analysis, and algebra into the field of mathematical analysis. Their mathematical precision is also pleasing. But they gain in unity and interest when presented in connection with boundary value problems.

The set of functions that make up the terms in the series representation is determined by the boundary value problem. Representations or expansions in Fourier series, certain types of series of sine or cosine functions, are associated with the more common boundary value problems. We shall give special attention to the theory and application of Fourier series. But we shall also consider extensions and generalizations of such series, including Fourier integrals and series of Bessel functions and Legendre polynomials.

A boundary value problem is correctly set if it has one and only one solution. Physical problems associated with partial differential equations often suggest boundary conditions under which a problem may be correctly set. In fact, it is sometimes helpful to interpret a problem physically in order to judge whether the boundary conditions may be adequate. This is a prominent reason for associating such problems with their physical applications, aside from the opportunity to display interesting and important contacts between mathematical analysis and the physical sciences.

The theory of partial differential equations gives results on the existence of solutions of boundary value problems. But such results are necessarily limited and complicated by the great variety of features: types of equations and conditions, and types of domains. Instead of appealing to general theory in treating a specific problem, we may actually find a solution and then prove that only one solution is possible.

2. Linear Boundary Value Problems. Theory and applications of ordinary or partial differential equations in a function u usually require that u satisfies not only the differential equation throughout some domain of its independent variable or variables but also some conditions on boundaries of that domain. The equations that represent those boundary conditions may involve values of derivatives of u, as well as u itself, at points on the boundary. In addition, some conditions on the continuity of u and its derivatives within the domain and at the boundaries are required.

Such a set of requirements constitutes a *boundary value problem* in the function u. We apply that term whenever the differential equation is accompanied by some boundary conditions even though the conditions may not be adequate to ensure a unique solution of the problem.

The three equations

$$\begin{aligned} &u''(x) - u(x) = -1 \qquad (0 < x < 1), \\ &u'(0) = 0, \qquad u(1) = 0, \end{aligned} \tag{1}$$

for example, constitute a boundary value problem in ordinary differential equations. The domain of the independent variable x is the interval $0 < x < 1$ whose boundaries consist of the two points $x = 0$ and $x = 1$. The solution of this problem which, together with each of its derivatives, is continuous everywhere is found to be

$$u(x) = 1 - (\cosh 1)^{-1} \cosh x. \tag{2}$$

Frequently it is convenient to indicate partial differentiation by writing independent variables as subscripts. If, for instance,

u is a function of the independent variables x and y, we may write

$$u_x \text{ or } u_x(x,y) \text{ for } \frac{\partial u}{\partial x}, \qquad u_{xx} \text{ for } \frac{\partial^2 u}{\partial x^2}, \qquad u_{xy} \text{ for } \frac{\partial^2 u}{\partial y\, \partial x},$$

and so on. Also, we shall be free to use the symbols $u_x(x_0,y)$ and $u_{xx}(x_0,y)$ to denote the values of the functions $\partial u/\partial x$ and $\partial^2 u/\partial x^2$, respectively, on the line $x = x_0$ and corresponding symbols for boundary values of other derivatives.

The problem consisting of the partial differential equation

$$u_{xx}(x,y) + u_{yy}(x,y) = 0 \qquad (x > 0, y > 0) \tag{3}$$

and the two boundary conditions

$$\begin{aligned} u(0,y) &= u_x(0,y) && (y > 0), \\ u(x,0) &= \sin x + \cos x && (x > 0), \end{aligned} \tag{4}$$

is an example of a boundary value problem in partial differential equations. The domain is the first quadrant of the xy plane. The reader can verify that the function

$$u(x,y) = e^{-y}(\sin x + \cos x) \tag{5}$$

is a solution of that problem. This function and its partial derivatives are everywhere continuous in the two variables x,y together and bounded in the domain $x > 0$, $y > 0$.

A differential equation in a function u, or a boundary condition on u, is *linear* if it is an equation of the *first degree in u and derivatives of u*. Thus the terms of the equation are either functions of the independent variables alone, including constants, or such functions multiplied by either u or one of the derivatives of u.

The differential equations and boundary conditions (1), (3), and (4) above are all linear. The differential equation

$$zu_{xx} + xy^2u_{yy} - e^z u_z = f(y,z) \tag{6}$$

in $u(x,y,z)$ is linear. But the equation

$$u_{xx} + uu_y = x$$

is nonlinear in $u(x,y)$ because the term uu_y is not of the first degree as an algebraic expression in the two variables u and u_y.

Let the letters A to G denote either constants or functions of the independent variables x and y only. Then the general *linear* *partial differential equation of second order in* $u(x,y)$ has the form

$$(7) \qquad Au_{xx} + Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu = G.$$

A boundary value problem is *linear* if its differential equation and all its boundary conditions are linear. Problem (1) and the problem consisting of equations (3) and (4) are examples of linear boundary value problems.

Methods of solution presented in this book do not apply to nonlinear boundary value problems.

A linear differential equation or boundary condition is *homogeneous* if each of its terms, other than zero itself, is of the first degree in the function u and its derivatives.

Equation (7) is homogeneous in a domain if and only if $G(x,y) = 0$ throughout that domain. Equation (6) is nonhomogeneous if $f(y,z) \neq 0$. Equation (3) and the first of conditions (4) are homogeneous. In our treatment of linear boundary value problems, homogeneous equations will play a distinctive role.

3. The Vibrating String. A tightly stretched string, whose position of equilibrium is some interval on the x axis, is vibrating in the xy plane. Each point of the string, with coordinates $(x,0)$ in the equilibrium position, has a transverse displacement $y(x,t)$ at time t. We assume that the displacements y are small relative to the length of the string, that slopes are small, and that other conditions are such that the movement of each point is essentially in the direction of the y axis. Then at time t the point has coordinates (x,y).

Let the tension P of the string be great enough that the string behaves as if it were perfectly flexible; that is, at each point the part of the string on the left of that point exerts the force of magnitude P in the tangential direction upon the part on the right; the effect of bending moments at the point can be neglected. The magnitude of the x component of the tensile force is denoted by H (Fig. 1). Our final assumption is that H is constant, that is, that the variation of H with x and t can be neglected.

Those idealizing assumptions are severe; but they are justified in many applications. They are adequately satisfied, for instance, by strings of musical instruments under ordinary conditions of operation. Mathematically, the assumptions lead to a partial differential equation in $y(x,t)$ which is linear.

Now let $V(x,t)$ denote the y component of the tensile force exerted by the left-hand portion of the string on the right-hand portion at the point (x,y). We take the positive sense of V as that of the y axis. If α is the slope angle of the string at the point (x,y) at time t, then $-V/H = \tan \alpha = \partial y/\partial x$ as indicated in Fig. 1. Thus *the y component V of the force exerted by the part of the string on the left of a point (x,y) upon the part on the right, at time t,* is given by the formula

$$V(x,t) = -Hy_x(x,t) \qquad (H > 0). \tag{1}$$

This is the basic formula for deriving the equation of motion of the string. It is also used in setting up certain types of boundary conditions.

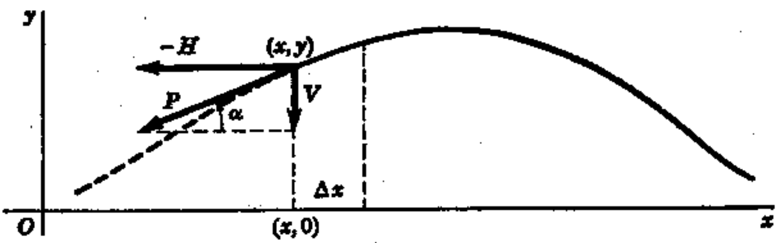

FIG. 1

Suppose that all external forces such as the weight of the string and resistance forces, which act on the string, other than forces at the end points, can be neglected. Consider a segment of the string not containing an end point, whose projection on the x axis has length Δx. Since x components of displacements are negligible, the mass of the segment is $\delta\,\Delta x$, where the constant δ is the mass of the string per unit length. At time t the y component of the force exerted by the string on the segment at the left-hand end (x,y) is $V(x,t)$, given by formula (1). The y component of the force exerted by the string on the other end of the segment is $-V(x + \Delta x,\, t)$, where the negative sign signifies that the force is exerted by the right-hand part upon the left-hand part at that point. The acceleration of the end (x,y) in the y direction is $y_{tt}(x,t)$. According to Newton's second law of motion (mass times acceleration equals force), then

$$\delta\,\Delta x\, y_{tt}(x,t) = -Hy_x(x,t) + Hy_x(x + \Delta x,\, t), \tag{2}$$

approximately, when Δx is small. Hence

$$y_{tt}(x,t) = \frac{H}{\delta} \lim_{\Delta x \to 0} \frac{y_x(x + \Delta x, t) - y_x(x,t)}{\Delta x} = \frac{H}{\delta} y_{xx}(x,t)$$

at each point where the partial derivatives exist.

Thus the function $y(x,t)$, representing the transverse displacements in a stretched string under the conditions stated above, satisfies the *wave equation*

$$\frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2} \qquad \left(a^2 = \frac{H}{\delta} > 0\right) \tag{3}$$

at points where no external forces act on the string. The constant a has the physical dimensions of velocity.

4. Modifications of the Equation. End Conditions. When external forces parallel to the y axis act along the string, let F denote the force per unit length of string. Then a term $F\,\Delta x$ must be added to the right-hand member of equation (2), Sec. 3, and the equation of motion is

$$y_{tt}(x,t) = a^2 y_{xx}(x,t) + F\delta^{-1}. \tag{1}$$

In particular, if the y axis is vertical with its positive sense upward and the external force consists of the weight of the string, then $F\,\Delta x = -\delta\,\Delta x\,g$, where the constant g is the acceleration of gravity. Equation (1) then becomes the linear nonhomogeneous equation

$$y_{tt}(x,t) = a^2 y_{xx}(x,t) - g. \tag{2}$$

In equation (1), F may be a function of x, t, y, or derivatives of y. In case the external force per unit length is a damping force proportional to the velocity in the y direction, for example, F is replaced by $-By_t$, where the positive constant B is a coefficient damping. Then the equation of motion is linear homogeneous:

$$y_{tt}(x,t) = a^2 y_{xx}(x,t) - by_t(x,t) \qquad (b = B\delta^{-1}). \tag{3}$$

If one end $x = 0$ of the string is kept fixed at the origin at all times $t \geqq 0$, the boundary condition at that end is clearly

$$y(0,t) = 0 \qquad (t \geqq 0). \tag{4}$$

But if that end is permitted to slide along the y axis and if the end is moved along that axis with a displacement $f(t)$, the boundary condition is the linear nonhomogeneous condition

$$y(0,t) = f(t) \qquad (t \geqq 0). \tag{5}$$

When the left-hand end is looped around the y axis and a force $g(t)$ in the y direction is applied to that end, $g(t)$ is the limit of the force $V(x,t)$ described in Sec. 3 as x tends to zero through positive values. The boundary condition is then

$$-Hy_x(0,t) = g(t) \qquad (t > 0). \tag{6}$$

The negative sign disappears if $x = 0$ is the right-hand end because $g(t)$ is then the force exerted on the part of the string to the left of that end.

5. Other Examples of Wave Equations. We can present further functions in physics and engineering which satisfy wave

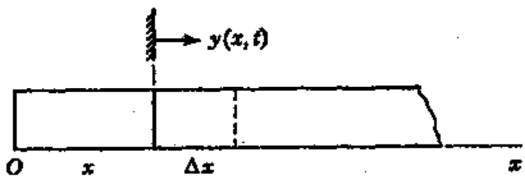

FIG. 2

equations and still limit our attention to fairly simple physical phenomena.

Longitudinal Vibrations of Bars. Let the coordinate x denote the distance from one end of an elastic bar in the shape of a cylinder or prism to other cross sections when the bar is unstrained. Displacements of the ends or initial displacements or velocities of the bar, all directed lengthwise along the bar and uniform over each cross section involved, cause the sections of the bar to move in the direction of the x axis. At time t the longitudinal displacement of the section labeled x is denoted by $y(x,t)$. Thus the origin of the displacement y of that section is fixed outside the bar, in the plane of the original reference position of that section (Fig. 2).

At the same time a neighboring section labeled $x + \Delta x$, to the right of section x, has a displacement $y(x + \Delta x, t)$; thus the element of the bar with natural length Δx is stretched by the amount $y(x + \Delta x, t) - y(x,t)$. Assuming that this extension or compression of the element satisfies Hooke's law, the force exerted upon the element over its left-hand end is, except for the effect of the inertia of the moving element,

$$-AE\,\frac{y(x+\Delta x,\,t)-y(x,t)}{\Delta x},$$

where A is the area of a cross section and E is the modulus of elasticity of the material in tension and compression. When Δx tends to zero, then it follows that the total longitudinal force $p(x,t)$ exerted on the section x by the part of the bar on the left of that section is given by the basic formula

$$p(x,t) = -AEy_x(x,t). \tag{1}$$

Let δ denote the mass of the material per unit volume. When we apply Newton's second law to the motion of an element of the bar of length Δx,

$$\delta A\,\Delta x\,y_{tt}(x,t) = -AEy_x(x,t) + AEy_x(x+\Delta x,\,t), \tag{2}$$

where the last term represents the force on the element at the end $x + \Delta x$, we find, after dividing by $\delta A\,\Delta x$ and letting Δx tend to zero, that

$$y_{tt}(x,t) = a^2 y_{xx}(x,t) \qquad (a^2 = E\delta^{-1}). \tag{3}$$

Thus the longitudinal displacements $y(x,t)$ in an elastic bar satisfy the wave equation (3) when no external longitudinal forces act on the bar other than at the ends. We have assumed only that displacements are small enough that Hooke's law applies and that sections remain plane after being displaced. The elastic bar here may be replaced by a column of air; then equation (3) has applications in the theory of sound.

The boundary condition $y(0,t) = 0$ signifies that the end $x = 0$ of the bar is held fixed. If instead the end $x = 0$ is free when $t > 0$, then no force acts across that end; that is, $p(0,t) = 0$ or, in view of formula (1),

$$y_x(0,t) = 0 \qquad (t > 0). \tag{4}$$

Transverse Vibrations of Membranes. Let $z(x,y,t)$ denote small displacements in the z direction, at time t, of points $(x,y,0)$ of a flexible membrane stretched tightly over a frame in the xy plane. The tensile stress P, the tension per unit length across any line on the membrane, is large, and the magnitude H of its component parallel to the xy plane is assumed to be constant. Then the internal force in the z direction at a section $x = x_0$, per unit length of that line, is $-Hz_x(x_0,y,t)$, corresponding to the force V (Sec. 3) in the vibrating string. The force in the z direction at a section $y = y_0$, per unit length, is $-Hz_y(x,y_0,t)$.

Consider an element of the membrane whose projection on the xy plane is a rectangle with opposite vertices $(x,y,0)$ and $(x + \Delta x, y + \Delta y, 0)$. When Newton's second law is applied to the motion of that element in the z direction, we find that $z(x,y,t)$ satisfies the two-dimensional wave equation

$$z_{tt} = a^2(z_{xx} + z_{yy}) \qquad (a^2 = H\delta^{-1}). \tag{5}$$

Here δ is the mass of the membrane per unit area. Details of the derivation are left to the problems.

If an external transverse force $F(x,y,t)$ per unit area acts over the membrane, the equation of motion takes the form

$$z_{tt} = a^2(z_{xx} + z_{yy}) + F\delta^{-1}. \tag{6}$$

PROBLEMS

1. Give details in the derivation of equation (1), Sec. 4, for the forced vibrations of a stretched string.

2. A tightly stretched string with its ends fixed at the points (0,0) and $(2c,0)$ hangs at rest under its own weight. The y axis points vertically upward. State why the static displacements $y(x)$ of points of the string satisfy the boundary value problem

$$a^2y''(x) - g = 0 \qquad (0 < x < 2c),$$
$$y(0) = y(2c) = 0.$$

Hence show that the string hangs in the parabolic arc

$$(x - c)^2 = \frac{2a^2}{g}\left(y + \frac{gc^2}{2a^2}\right) \qquad (0 \leqq x \leqq 2c).$$

Show that the depth of the vertex of the arc varies directly with δ and c^2 and inversely with H.

3. Use formula (1), Sec. 3, for the vertical force V and the formula for y in Problem 2 to show that the vertical force exerted on that string by either support is $g\delta c$, half the weight of the string.

4. A strand of wire 1 ft long, stretched between the origin and the point (1,0) with tension $H = 10$ lb, weighs 0.032 lb ($g\delta = 0.032$, $g =$

32 ft/sec²). At the instant $t = 0$ the strand lies along the x axis, but it has a velocity 1 ft/sec in the direction of the y axis, perhaps because the supports were in motion and were brought to rest at that instant. If no external forces act along the wire, show why the displacements $y(x,t)$ should satisfy this boundary value problem:

$$y_{tt}(x,t) = 10^4 y_{xx}(x,t) \qquad (0 < x < 1,\ t > 0),$$
$$y(0,t) = y(1,t) = 0, \qquad y(x,0) = 0, \qquad y_t(x,0) = 1.$$

5. The physical dimensions of the force H, the tension in the string, are those of mass times acceleration, MLT^{-2}, where M denotes mass, L length, and T time. Since $a^2 = H\delta^{-1}$, show that a has the dimensions of velocity LT^{-1}.

6. The end $x = 0$ of a cylindrical elastic bar is kept fixed, and a constant compressive force of magnitude F_0 units per unit area is exerted at all times $t > 0$ over the end $x = c$. If the bar is initially unstrained and at rest and if no external forces act along the bar, verify that the function $y(x,t)$ representing the longitudinal displacements of cross sections should satisfy this boundary value problem:

$$y_{tt}(x,t) = a^2 y_{xx}(x,t) \qquad (0 < x < c,\ t > 0;\ a^2 = E\delta^{-1})$$
$$y(0,t) = 0, \qquad Ey_x(c,t) = -F_0, \qquad y(x,0) = y_t(x,0) = 0.$$

7. The left-hand end $x = 0$ of an elastic bar is elastically supported in such a way that the longitudinal force per unit area exerted on the bar at that end is proportional to the displacement of the end, but opposite in sign. Show that the end condition there has the form

$$Ey_x(0,t) = Ky(0,t) \qquad (K > 0).$$

8. Derive equation (6), Sec. 5. Also note that the *static transverse displacements* $z(x,y)$ of a membrane, over which a transverse force $F(x,y)$ per unit area acts, satisfy *Poisson's equation*

$$z_{xx} + z_{yy} + k = 0 \qquad (k = FH^{-1}).$$

6. Conduction of Heat. Thermal energy is transferred from warmer to cooler regions interior to a solid body by conduction. It is convenient to refer to that transfer as the flow of heat, as if heat were a fluid or gas which diffuses through the body from regions of high concentration into regions of low concentration of that fluid.

Let P_0 denote a point (x_0,y_0,z_0) interior to the body and S a plane or smooth curved surface through P_0. At a time t_0 the *flux* $\Phi(x_0,y_0,z_0,t_0)$ *of heat* across S at P_0 is the quantity of heat per unit area, per unit time, that is being conducted across S at that