PROBABILISTIC THEORY OF STRUCTURAL DYNAMICS

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PREFACE

"Probabilistic Theory of Structural Dynamics" is the outgrowth of lecture notes for a graduate course entitled Stochastic Structural Dynamics that I have taught at the University of Illinois since 1961. Hence, one of the purposes of this book is to serve as a textbook for graduate students in structural engineering. As a structural engineering textbook in random vibrations—as the general field is now more widely known—it emphasizes analyses of the responses of practical structures, such as beams, plates, and their combinations, to random excitations of known probability descriptions or known statistical properties. The number of examples introduced herein make this book also suitable as a reference for research workers.

Briefly, the organization is as follows: Chapter 1 is an explanation of the scope of the probabilistic theory of structural dynamics. Chapters 2 through 4 present the elements of probability theory necessary to the subsequent structural analyses. Chapters 5 through 8 consider in turn the random vibration of single-degree-of-freedom, multiple-degrees-of-freedom, continuous, and nonlinear structures. Chapter 9 discusses structural reliability and related topics. A knowledge of calculus, differential equations, matrices, strength of materials, and mechanical vibrations is the prerequisite for the complete use of this book.

It is a pleasure to acknowledge the help I received during the preparation of the manuscript. Among my colleagues at the Aeronautical and Astronautical Engineering Department of the University of Illinois, I am indebted to Prof. H. S. Stillwell, head of the department, for giving me warm and constant encouragement and to Prof. H. H. Hilton, who convinced me of the need for a course in stochastic structural dynamics at the University of Illinois. I am grateful for the invaluable and comprehensive suggestions and criticisms of Prof. S. H. Crandall of the Massachusetts Institute of Technology, who reviewed the manuscript. Among my students, my special thanks are due B. K. Donaldson, who painstakingly checked over the entire manuscript and with whom I have had many stimulating discussions, and T. J. McDaniel, who ably assisted in a number of computations. I also wish to thank Miss Dorothy Nugent and her staff for their unfailing assistance in putting the manuscript in proper order, and especially Mrs. R. E. Richardson, who typed the major part of the manuscript.

Y. K. LIN

CONTENTS

1 | INTRODUCTION

The realm of structural dynamics has been considerably enlarged since the introduction of probabilistic methods. Previously, the structural engineer would always cope with design needs by use of a deterministic analysis. In such an analysis, he would assume a complete knowledge of the dynamic properties and the initial state of a structure and the exact time history of its excitation. Consider, for example, the simplest situation in which a structure is idealized as a spring-mass-damper system excited by an external force acting on the mass, as shown in Fig. 1.1. If at time $t = 0$, the displacement and velocity of the mass are, respectively,

$$x(0) = x_0 \qquad \dot{x}(0) = v_0 \tag{1-1}$$

then the general expression for the displacement $x(t)$ may be written as follows:

$$x(t) = g(x_0, v_0, m, k, c, t) + \int_0^t f(\tau)h(m, k, c, t, \tau)\, d\tau \tag{1-2}$$

If it is assumed that the value of every element on the right-hand side of Eq. (1-2) is precisely known, then the motion of the mass $x(t)$ can be computed exactly. In a probabilistic analysis we admit uncertainty of knowledge of one or several elements on the right-hand side of Eq. (1-2). For the sake of discussion, let it be uncertainty about the spring rate. We shall call an uncertain quantity *random*. If we pick and carefully calibrate one particular spring, we cannot help but find that it has a definite extension-contraction rate. Therefore, as far as one particular spring is concerned, the probabilistic viewpoint is not appropriate, or at most it is only trivial. Suppose, however, that we are interested in any such spring manufactured from a certain process. Since not all springs

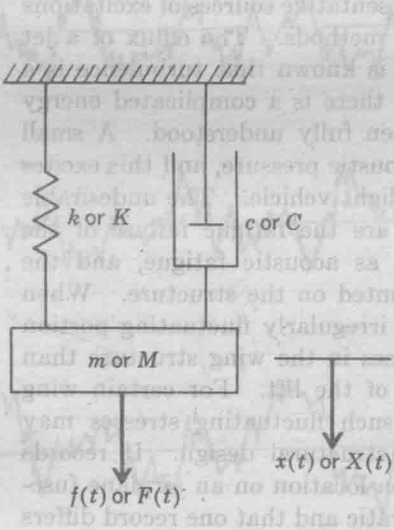


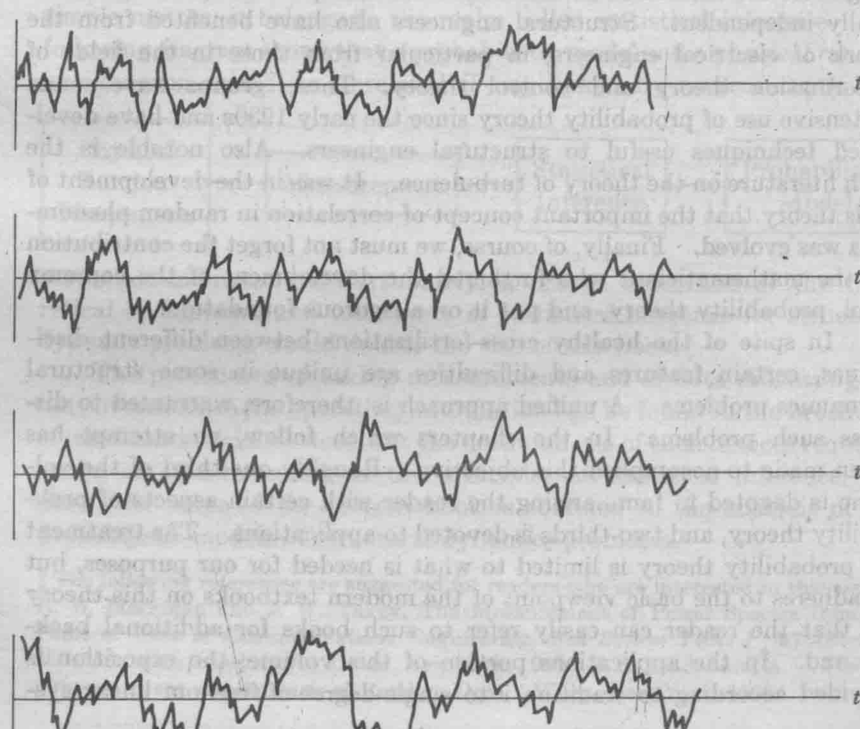
Fig. 1.1 Spring-mass-damper system. Small letters denote deterministic quantities; capital letters denote random quantities.

so manufactured are identical, before one sample is picked from a batch and tested it is not possible to foretell the exact spring rate. From this example we see that the adjective *random* implies a collection of samples none of which has been separated from the others, and that once an individual sample has been taken from the collection, it is deterministic. Similarly, when we say that the forcing function is random, we mean that the analysis is for a collection of exciting forces, each one of which is generally different from the others, but any one of which can be encountered in the actual service of the structure. Obviously, when one or more of the elements of the right-hand side of Eq. (1-2) are random, the displacement must also be random. It is possible, however, to deduce the random nature of the displacement from the random nature of the spring rate, the excitation, etc.

Since no material is perfectly homogeneous, no beam is perfectly uniform, and no rivet perfectly fits a hole, etc., it is clear that the probabilistic viewpoint is more realistic, although in those cases where the uncertainties involved are small and are not the major issue, the simpler deterministic approach may well be satisfactory. For example, by adequate specification and supervision, the fabrication of a structure may be controlled so that the uncertainty about the dynamic properties of the structure may be overlooked. The uncertainty about the excitation is generally greater, although sometimes a wise choice of a representative load or the choice of the most severe load can be made and the consequent deterministic analysis may lead to a useful design although perhaps not to the optimum design. Nevertheless, the primary incentive for the adaptation of probabilistic methods in structural dynamics analyses has been random excitations. In aerospace engineering applications, jet or rocket engine noise and gusts are two representative sources of excitations which should be treated by probabilistic methods. The efflux of a jet or rocket engine is a turbulent flow. It is known that turbulence is a phenomenon of flow instability and that there is a complicated energy exchange taking place which has not been fully understood. A small portion of the energy is converted into acoustic pressure, and this excites the nearby structural component of the flight vehicle. The undesirable consequences of this type of excitation are the fatigue failure of the nearby structure, now generally known as acoustic fatigue, and the malfunction of electronic equipment mounted on the structure. When an airplane flies into a gusty region, the irregularly fluctuating portion of the total lift can produce higher stresses in the wing structure than those resulting from the steady portion of the lift. For certain wing configurations, it has been found that such fluctuating stresses may become the major consideration in the structural design. If records were taken of jet noise pressure at a given location on an airplane fuselage, we should find that they are very erratic and that one record differs from another. The same unpredictability and lack of resemblance to each other are also characteristics of the records of gust velocity. Figure 1.2 depicts some such records placed side by side. It is easy to see that this type of physical phenomenon must be regarded as being random.

Examples of excitations which are essentially random and which act upon other than aerospace structures are plentiful. Earthquake, blast, and wind loads acting on architectural structures fall into this general category. The excitation experienced by a ship hull in a confused sea is similar to that of an airplane in a gusty atmosphere. Nevertheless, it has been primarily the applications in aerospace engineering which have speeded the development of the probabilistic techniques in the area of structural dynamics. It is worth mentioning that these techniques are being accepted more and more by the government purchasing and certifying agencies as being more suitable for the analysis of the structural integrity of a product. It may not be too long before probabilistic design requirements for earthquake, blast, and wind loads will be incorporated in most building codes.

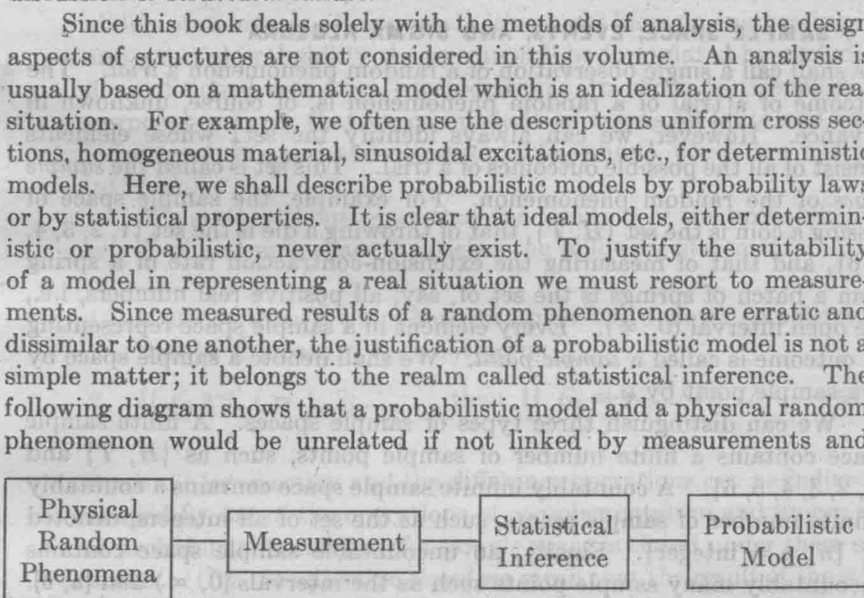
Fig. 1.2 Typical records of a random phenomenon.



Although structural applications have provided the immediate incentive for developing the probabilistic theory of structural dynamics on a broad base, it is well to mention that structural engineers have inherited a considerable amount of knowledge from the early work of physicists on the subject of Brownian motion. The first paper treating Brownian motion as a random process was published by Einstein in 1905. It is interesting to note that Einstein's problem may be considered as a special case of the spring-mass-damper system with both the spring rate and the mass of the system negligibly small. Later, Einstein's work was extended by many others, notably Ornstein (1917), Uhlenbeck and Ornstein (1930), Van Lear and Uhlenbeck (1931), and Wang and Uhlenbeck (1945). However, from the standpoint of Brownian motion, one is generally more interested in the behavior of closed systems, closed in the sense that either the simple mass particle or the complicated continuum, say a beam, is surrounded by a fluid medium which is in a state of so-called statistical equilibrium. The main objective is to study how the Brownian motion of the mass particle or the beam tends to statistical equilibrium. Since, for such cases, the excitation and the dissipation forces are both provided by the fluid medium, the two types of force are related. On the other hand, structural engineers are interested in excitations and dissipations which are essentially independent. Structural engineers also have benefited from the work of electrical engineers, in particular from those in the fields of information theory and control theory. These groups have made extensive use of probability theory since the early 1930s and have developed techniques useful to structural engineers. Also notable is the rich literature on the theory of turbulence. It was in the development of this theory that the important concept of correlation in random phenomena was evolved. Finally, of course, we must not forget the contribution of the mathematicians who furthered the development of the common tool, probability theory, and put it on a rigorous foundation.

In spite of the healthy cross-fertilizations between different disciplines, certain features and difficulties are unique in some structural dynamics problems. A unified approach is, therefore, warranted to discuss such problems. In the chapters which follow, an attempt has been made to accomplish this objective. Roughly one-third of the volume is devoted to familiarizing the reader with certain aspects of probability theory, and two-thirds is devoted to applications. The treatment of probability theory is limited to what is needed for our purposes, but it adheres to the basic viewpoint of the modern textbooks on this theory so that the reader can easily refer to such books for additional background. In the applications portion of this volume, the exposition is divided according to tradition into single-degree-of-freedom linear systems, multiple-degrees-of-freedom linear systems, continuous linear systems, and nonlinear systems. In all cases the dynamic characteristics of a system are assumed to be deterministic, and the excitation is assumed to be random. The objective is to compute the probability law which governs the behavior of the random response of the system or, in various orders of completeness, statistical properties possessed by the response. However, from an engineering standpoint the final goal is the ability to make statements regarding the reliability of a structure to withstand random excitations. Therefore, the concluding chapter is devoted to a discussion of structural failure.

Since this book deals solely with the methods of analysis, the design aspects of structures are not considered in this volume. An analysis is usually based on a mathematical model which is an idealization of the real situation. For example, we often use the descriptions uniform cross sections, homogeneous material, sinusoidal excitations, etc., for deterministic models. Here, we shall describe probabilistic models by probability laws or by statistical properties. It is clear that ideal models, either deterministic or probabilistic, never actually exist. To justify the suitability of a model in representing a real situation we must resort to measurements. Since measured results of a random phenomenon are erratic and dissimilar to one another, the justification of a probabilistic model is not a simple matter; it belongs to the realm called statistical inference. The following diagram shows that a probabilistic model and a physical random phenomenon would be unrelated if not linked by measurements and



statistical inference. Unfortunately, a proper exposition of the theoretical background and techniques of statistical inference for structural dynamic problems would double the size of this book.‡

The procedures of taking measurements and of data processing are not covered since the operating information can be found in the brochures of most commercial devices; but the optimum use of such devices requires, again, an understanding of the principles of statistical inference. In short, the scope of the present book is confined to the analysis of the probabilistic models of structural dynamics problems.

‡ The following references are suggested for readers who are interested in this aspect. R. B. Blackman and J. W. Tukey, The Measurement of Power Spectra from the Point of View of Communications Engineering, *Bell System Tech. J.*, **37**:185–282, 485–569 (1958); reprinted by Dover, New York, 1959. J. S. Bendat and A. G. Piersol, "Measurement and Analysis of Random Data," Wiley, New York, 1966.

2 | SOME ASPECTS OF PROBABILITY THEORY

It is not possible nor necessary to give a thorough exposition here of modern probability theory. Chapters 2 through 4 will sketch, however, certain aspects of this theory required for our study of the probabilistic theory of structural dynamics, for the sake of being self-contained as well as for building a common language for later discussion.

2.1 / SAMPLE SPACE, EVENTS, AND SIGMA ALGEBRA

We shall call a single observation of a random phenomenon a *trial*. The outcome of a trial of a random phenomenon is, of course, unknown in advance. However, we can always identify the set‡ whose elements consist of all the possible outcomes of a trial. This set is called the *sample space* of the random phenomenon. For example, the sample space of tossing a coin is the set $\{H, T\}$, that of throwing a die is the set $\{1, 2, 3, 4, 5, 6\}$, and that of measuring the extension-contraction rate of a spring from a batch of springs is the set of, say, all positive real numbers, i.e., the open interval $(0, \infty)$. Every element in a sample space representing an outcome is called a *sample point*. We shall denote a sample space by Ω, a sample point by ω.

We can distinguish three types of sample spaces. A finite sample space contains a finite number of sample points, such as $\{H, T\}$ and $\{1, 2, 3, 4, 5, 6\}$. A countably infinite sample space contains a countably infinite number of sample points, such as the set of all integers, denoted by $\{n: n = \text{integer}\}$. Finally, an uncountable sample space contains uncountably many sample points such as the intervals $[0, \infty)$ and $[a, b]$.

Next, we introduce the concept of an *event*. Suppose that the outcome of a trial of throwing a die is 4. We may say that several events have occurred: (*a*) number 4, (*b*) an even number, (*c*) a number greater than 2, (*d*) a number smaller than 5, (*e*) a number equal to or smaller than 6, etc. Note that any of the events (*b*) through (*e*) may also occur when the outcome is not 4. Since the sample space contains all the outcomes, every event is a subset of the sample space. Event (*a*), containing only a single sample point, is called an *elementary* event; and events (*b*) through (*e*), each containing more than one sample point, are called *compound* events. An event is called a *certain* event when it contains *all* the elements in the sample space. Thus, event (*e*) is a certain event. Opposite to a certain event is an *impossible* event which contains *no* sample point. In the present example, the event a number smaller than 1 is impossible. Regarding both a certain event and an impossible event

‡ Readers unfamiliar with the terminology and the algebra of simple set theory may consult Appendix IV.

as subsets of the sample space is consistent with the usual practice in set theory of regarding a set as a subset of itself, and the empty set as a subset of any set.

Although every event is a subset of the sample space, it is not always true that every subset of the sample space is an event. We postulate that an event must be one for which a probability of occurrence can be specified. It is known in advanced probability theory that by use of certain complicated limiting procedures we can obtain some subsets of an uncountable sample space about which we cannot make consistent probability statements. Fortunately, these sets have no engineering interest, and a useful probability theory can still be formulated by excluding such ill-conditioned sets from consideration. In short, it is enough for our purposes to know that we can include as events all finite and countably infinite subsets and those uncountable subsets which are composed of intervals.

We state without proof that the family $\mathfrak{F}$ of the probabilizable subsets E of a sample space Ω satisfies the following statements:

1. If $E \in \mathfrak{F}$, then $\bar{E} \in \mathfrak{F}$, where an upper bar denotes complementation; that is, $\bar{E}$ means E does not occur.

2. If $E_i \in \mathfrak{F}$, $i = 1, 2, \ldots$, then $\bigcup_{i=1}^{\infty} E_i \equiv E_1 \cup E_2 \cup \cdots \in \mathfrak{F}$.

Since the intersection and the difference operations can be indirectly performed by use of the operations of complementation and union, a family of probabilizable subsets of a sample space is closed under these set operations. (By the adjective *closed* we mean that the result of the set operation on the members of the family $\mathfrak{F}$ must also belong to this family.) A family of subsets of a set which satisfies statements 1 and 2 is called a *sigma algebra*.

2.2 / AXIOMS AND SOME THEOREMS OF PROBABILITY THEORY

With the above background, we can now state the following axioms of probability theory:

Axiom 1. For each member E belonging to a sigma algebra $\mathfrak{F}$

$$0 \le \mathcal{P}(E) \le 1$$

Here $\mathcal{P}(E)$ denotes the probability of occurrence of the event E. It reads "the probability measure of E," or simply "the probability of E."

Axiom 2. $\mathcal{P}(\Omega) = 1$.

Axiom 3. If $E_1, E_2, \ldots, E_n$ are mutually exclusive, then

$$\mathcal{P}\left(\bigcup_{j=1}^{n} E_j\right) = \sum_{j=1}^{n} \mathcal{P}(E_j)$$

where n may be finite or infinite.

A few observations are worth special attention. We first note that a probability measure $\mathcal{P}$ is a *set function*, since its argument is generally a set which may contain more than one point. Secondly, the probability associated with the entire sample space Ω, a certain event, is unity; thus the assignment of probability measures to different events is analogous to the distribution of a unit mass over the sample space. For this reason some authors have aptly called $\mathcal{P}$ the probability mass function. Thirdly, in order that Axiom 3 may be satisfied, the probabilities assigned to individual sample points in the case of an uncountable sample space must be either all zero or nearly all zero except possibly for a subset of countable sample points in the sample space. The third observation is not surprising if we compare this situation with the distribution of a finite load over a continuous structure, say a flat plate. If the total load is limited to unity, then there can be at most a countable number of concentrated loads acting on the plate. The nonconcentrated portion of the total load is distributed in a prescribed manner over the surface of the plate. Since a single point occupies a *zero* area on the surface, it is subjected to a zero load if not directly under one of the concentrated loads.

Theorems and definitions

1. Theorem of the Complementary Event.

$$\mathcal{P}(\bar{E}) = 1 - \mathcal{P}(E) \tag{2-1}$$

Corollary:

$$\mathcal{P}(\phi) = 0 \tag{2-2}$$

where $\phi = \bar{\Omega}$ represents nothing happens, an impossible event.

2. Theorem of the Total Event.

$$\mathcal{P}(E_1 \cup E_2) = \mathcal{P}(E_1) + \mathcal{P}(E_2) - \mathcal{P}(E_1 \cap E_2) \tag{2-3}$$

Note that $E_1 \cup E_2$ is the occurrence of either E_1 or E_2, and $E_1 \cap E_2$ is the occurrence of both E_1 and E_2.

Corollary:

$$\begin{aligned}\mathcal{P}(E_1 \cup E_2 \cup \cdots \cup E_n) = \Sigma\mathcal{P}(E_i) - \Sigma\mathcal{P}(E_i \cap E_j) \\ + \Sigma\mathcal{P}(E_i \cap E_j \cap E_k) - \cdots \\ \pm (-1)^{n-1}\mathcal{P}(E_1 \cap E_2 \cap \cdots \cap E_n)\end{aligned} \tag{2-4}$$

where each summation includes all distinct combinations of distinct events.

3. Definition of Conditional Probability. The probability of the occurrence of event E_2 conditional upon the occurrence of event E_1, denoted by $\mathcal{P}(E_2|E_1)$, is defined as

$$\mathcal{P}(E_2|E_1) = \frac{\mathcal{P}(E_1 \cap E_2)}{\mathcal{P}(E_1)} \quad \text{if } \mathcal{P}(E_1) > 0 \tag{2-5}$$

If $\mathcal{P}(E_1) = 0$, the conditional probability is undefined.

4. Definition of Independence. Event E_2 is said to be independent of event E_1 if

$$\mathcal{P}(E_2|E_1) = \mathcal{P}(E_2) \tag{2-6}$$

5. Theorem of Joint Events.

$$\begin{aligned}\mathcal{P}(E_1 \cap E_2) &= \mathcal{P}(E_1)\mathcal{P}(E_2|E_1) \\ &= \mathcal{P}(E_2)\mathcal{P}(E_1|E_2)\end{aligned} \tag{2-7}$$

$$\mathcal{P}(E_1 \cap E_2 \cap \cdots \cap E_n) = \mathcal{P}(E_1)\mathcal{P}(E_2|E_1)\mathcal{P}(E_3|E_1 \cap E_2) \cdots \mathcal{P}(E_n|E_1 \cap E_2 \cap \cdots \cap E_{n-1}) \tag{2-8}$$

There are $n!$ ways of expressing the same joint probability in (2-8) corresponding to the different permutation of $E_1, E_2, \ldots, E_n$.

Corollary:

If $E_1, E_2, \ldots, E_n$ are independent one with the others, then

$$\mathcal{P}(E_1 \cap E_2 \cap \cdots \cap E_n) = \mathcal{P}(E_1)\mathcal{P}(E_2) \cdots \mathcal{P}(E_n) \tag{2-9}$$

Although the probability of a certain event is 1 and the probability of an impossible event is zero, the reverse is not necessarily true; that is, an event with probability 1 is not always certain, nor is an event with probability zero always impossible. This observation becomes clear if we recall that generally we can only assign zero probability to a sample point in a uncountable sample space. It is also important to note that Eq. (2-9) may be true without the events E_i being independent of each other as a system. For all the events E_i to be independent, all the following equations would have to be satisfied:

$$\begin{aligned}&\mathcal{P}(E_i \cap E_j) = \mathcal{P}(E_i)\mathcal{P}(E_j) \\ &\mathcal{P}(E_i \cap E_j \cap E_k) = \mathcal{P}(E_i)\mathcal{P}(E_j)\mathcal{P}(E_k) \\ &\cdots\cdots\cdots \\ &\mathcal{P}(E_1 \cap E_2 \cap E_3 \cap \cdots \cap E_n) = \mathcal{P}(E_1)\mathcal{P}(E_2)\mathcal{P}(E_3) \cdots \mathcal{P}(E_n)\end{aligned} \tag{2-10}$$

However, when only two events are considered, then

$$\mathcal{P}(E_1 \cap E_2) = \mathcal{P}(E_1)\mathcal{P}(E_2)$$

will lead to both

$$\mathcal{P}(E_1|E_2) = \mathcal{P}(E_1) \quad \text{and} \quad \mathcal{P}(E_2|E_1) = \mathcal{P}(E_2)$$

and the conclusion that they are truly mutually independent.

2.3 / STATISTICAL REGULARITY

The foregoing sections have been devoted to the development of a modern concept of probability without being unduly rigorous. So far we have remained unconcerned with physical realities. In particular, the question of how to assign probabilities to physical events has been untouched.

It is not difficult to determine at least the general form of the probability measure $\mathcal{P}$ for a simple random experiment such as flipping a coin or tossing a die. In the example of flipping a coin, it is obvious that $\mathcal{P}(H) = \mathcal{P}(T) = \frac{1}{2}$ if the coin is fair. Even when the coin is not fair we can still write $\mathcal{P}(H) = p$ and $\mathcal{P}(T) = q = 1 - p$, and proceed to determine the more interesting probability for the number of H's (or T's) obtained from flipping the same coin N times. The die-tossing problem is similar. Both problems are treated extensively in elementary books on probability theory. It seems reasonable to assume, however, that readers of this book are not interested merely in flipping coins or tossing dice. An immediate question in each mind more likely will be, How can probabilities be determined for events associated with more complicated engineering problems? Unfortunately, nature is generally reluctant to reveal the exact probabilistic mechanism of a physical phenomenon, and man has to exercise his best judgment based upon some available clues.

Our daily experiences show that repeated trials of a random experiment exhibit a certain regularity such that averages of the outcomes tend to recognizable limits when the number of trials becomes large. This tendency is called *statistical regularity*. It is important to note that we speak of statistical regularity for each random experiment; i.e., the trials are repeated under an identical set of conditions. Obviously, it would be fruitless to attempt to establish any regularity for trials under different sets of conditions.

Let N be the total number of trials of a random experiment, and N_E the number of occurrences of an event E. We define the relative frequency of the event E as

$$r_N(E) = \frac{N_E}{N} \tag{2-11}$$

One version of statistical regularity is stated as follows:

$$\lim_{N\to\infty} \mathcal{P}\{|r_N(E) - p| \geq \epsilon\} = 0 \tag{2-12}$$