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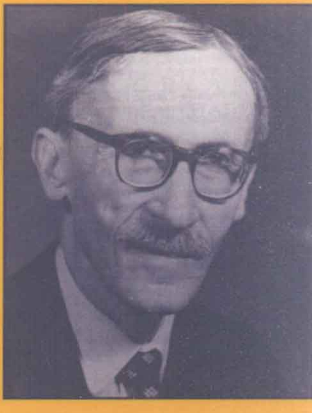
Lévy Matters I

A subseries on Lévy Processes



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Ken-iti Sato • Christoph Schwab

Lévy Matters I

Recent Progress in Theory and Applications: Foundations, Trees and Numerical Issues in Finance

With a Short Biography of Paul Lévy
by Jean Jacod

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Preface

Over the past 10-15 years, we have seen a revival of general Lévy processes theory as well as a burst of new applications. In the past, Brownian motion or the Poisson process have been considered as appropriate models for most applications. Nowadays, the need for more realistic modelling of irregular behaviour of phenomena in nature and society like jumps, bursts, and extremes has led to a renaissance of the theory of general Lévy processes. Theoretical and applied researchers in fields as diverse as quantum theory, statistical physics, meteorology, seismology, statistics, insurance, finance, and telecommunication have realised the enormous flexibility of Lévy models in modelling jumps, tails, dependence and sample path behaviour. Lévy processes or Lévy driven processes feature slow or rapid structural breaks, extremal behaviour, clustering, and clumping of points.

Tools and techniques from related but disctinct mathematical fields, such as point processes, stochastic integration, probability theory in abstract spaces, and differential geometry, have contributed to a better understanding of Lévy jump processes.

As in many other fields, the enormous power of modern computers has also changed the view of Lévy processes. Simulation methods for paths of Lévy processes and realisations of their functionals have been developed. Monte Carlo simulation makes it possible to determine the distribution of functionals of sample paths of Lévy processes to a high level of accuracy.

This development of Lévy processes was accompanied and triggered by a series of Conferences on Lévy Processes: Theory and Applications. The First and Second Conferences were held in Aarhus (1999, 2002), the Third in Paris (2003), the Fourth in Manchester (2005), and the Fifth in Copenhagen (2007).

To show the broad spectrum of these conferences, the following topics are taken from the announcement of the Copenhagen conference:

- Structural results for Lévy processes: distribution and path properties
- Lévy trees, superprocesses and branching theory
- Fractal processes and fractal phenomena
- Stable and infinitely divisible processes and distributions
- Applications in finance, physics, biosciences and telecommunications
- Lévy processes on abstract structures
- Statistical, numerical and simulation aspects of Lévy processes
- Lévy and stable random fields.

At the Conference on Lévy Processes: Theory and Applications in Copenhagen the idea was born to start a series of Lecture Notes on Lévy processes to bear witness of the exciting recent advances in the area of Lévy processes and their applications. Its goal is the dissemination of important developments in theory and applications. Each volume will describe state of the art results of this rapidly evolving subject with special emphasis on the non-Brownian world. Leading experts will present new exciting fields, or surveys of recent developments, or focus on some of the most promising applications. Despite its special character, each article is written in an expository style, normally with an extensive bibliography at the end. In this way each article makes an invaluable comprehensive reference text. The intended audience are PhD and postdoctoral students, or researchers, who want to learn about recent advances in the theory of Lévy processes and to get an overview of new applications in different fields.

Now, with the field in full flourish and with future interest definitely increasing it seemed reasonable to start a series of Lecture Notes in this area. The present volume is the first in the series, and future volumes will appear over time under the common name "Lévy Matters", in tune with the developments in the field. "Lévy Matters" will appear as a subseries of the Springer Lecture Notes in Mathematics, thus ensuring wide dissemination of the scientific material. The expository articles in this first volume have been chosen to reflect the broadness of the area of Lévy processes.

The first article by Ken-iti Sato characterises extensions of the class of selfdecomposable distributions on $\mathbb{R}^d$. They are given as two families each with two continuous parameters of classes of distributions of improper stochastic integrals $\lim_{t\to\infty}\int_0^t f(s)dX_s$ for appropriate non-random functions f and Lévy processes X. Many known classes appear as limiting cases in some parameters: the Thorin class, the Goldie-Steutel-Bondesson class, and the class of completely selfdecomposable distributions. Moreover, the theory of fractional integrals of measures is built.

The second article by Thomas Duquesne discusses Hausdorff and packing measures of stable trees. Stable trees are a special class of Lévy trees, which form a class of random compact metric spaces, and were introduced by Le Gall and Le Jan (1998) as the genealogy of continuous state branching processes. It is shown that level sets of stable trees are the sets of points situated at a given distance from the root. In contrast to Brownian trees, for non-Brownian stable trees there is no exact packing measure for level sets, i.e. the sets of points situated at a given distance from the root.

The third (and last) article by Oleg Reichmann and Christoph Schwab presents numerical solutions to Kolmogorov equations, which arise for instance in financial engineering, when Lévy or additive processes model the dynamics of the risky assets. Solution algorithms based on wavelet representations for the Dirichlet and free boundary problems connected to barrier and American style contracts are presented. Lévy copulas are used for a systematic construction of parametric multivariate Feller-Lévy processes. Numerical aspects of the implementation and Monte Carlo path simulation techniques are addressed.

We take the possibility to acknowledge the very positive collaboration with the relevant Springer staff and the Editors of the LN Series, and the (anonymous) referees of the three articles.

We hope that the readers of this and subsequent volumes enjoy learning about the high potential of Lévy processes in theory and applications. Researchers with ideas for contributions to further volumes in the Lévy Matters series are invited to contact any of the Editors with proposals or suggestions.

June 2010

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A Short Biography of Paul Lévy

The first volume of the series "Lévy Matters" would not be complete without a short sketch about the life and mathematical achievements of the mathematician whose name has been borrowed and used here. This is more a form of tribute to Paul Lévy, who not only invented what we call now Lévy processes, but also is in a sense the founder of the way we are now looking at stochastic processes, with emphasis on the path properties.

Paul Lévy was born in 1886, and lived until 1971. He studied at the Ecole Polytechnique in Paris, and was soon appointed as professor of mathematics in the same institution, a position that he held from 1920 to 1959. He started his career as an analyst, with 20 published papers between 1905 (he was then 19 years old) and 1914, and he became interested in probability by chance, so to speak, when asked to give a series of lectures on this topic in 1919 in that same school: this was the starting point of an astounding series of contributions in this field, in parallel with a continuing activity in functional analysis.

Very briefly, one can mention that he is the mathematician who introduced characteristic functions in full generality, proving in particular the characterisation theorem and the first "Lévy's theorem" about convergence. This naturally led him to study more deeply the convergence in law with its metric, and also to consider sums of independent variables, a hot topic at the time: Paul Lévy proved a form of the 0-1 law, as well as many other results, for series of independent variables. He also introduced stable and quasi-stable distributions, and unravelled their weak and/or strong domains of attractions, simultaneously with Feller.

Then we arrive at the book "Théorie de l'addition des variables aléatoires", published in 1937, and in which he summarizes his findings about what he called "additive processes" (the homogeneous additive processes are now called Lévy processes, but he did not restrict his attention to the homogeneous case). This book contains a host of new ideas and new concepts: the decomposition into the sum of jumps at fixed times and the rest of the process; the Poissonian structure of the jumps for an additive process without fixed times of discontinuities; the "compensation" of those jumps so that one is able to sum up all of them; the fact that the remaining continuous part is Gaussian. As a consequence, he implicitly gave the formula providing the form of all additive processes without fixed discontinuities, now called the Lévy-Itô Formula, and he proved the Lévy-Khintchine formula for the characteristic functions of all infinitely divisible distributions. But, as fundamental as all those results are, this book contains more: new methods, like martingales which, although not given a name, are used in a fundamental way; and also a new way of looking at processes, which is the "pathwise" way: he was certainly the first to understand the importance of looking at and describing the paths of a stochastic process, instead of considering that everything is encapsulated into the distribution of the processes.

This is of course not the end of the story. Paul Lévy undertook a very deep analysis of Brownian motion, culminating in his book "Processus stochastiques et mouvement brownien" in 1948, completed by a second edition in 1965. This is a remarkable achievement, in the spirit of path properties, and again it contains so many deep results: the Lévy modulus of continuity, the Hausdorff dimension of the path, the multiple points, the Lévy characterisation theorem. He introduced local time, proved the arc-sine law. He was also the first to consider genuine stochastic integrals, with the area formula. In this topic again, his ideas have been the origin of a huge amount of subsequent work, which is still going on. It also laid some of the basis for the fine study of Markov processes, like the local time again, or the new concept of instantaneous state. He also initiated the topic of multi-parameter stochastic processes, introducing in particular the multi-parameter Brownian motion.

As should be quite clear, the account given here does not describe the whole of Paul Lévy's mathematical achievements, and one can consult for many more details the first paper (by Michel Loève) published in the first issue of the Annals of Probability (1973). It also does not account for the humanity and gentleness of the person Paul Lévy. But I would like to end this short exposition of Paul Lévy's work by hoping that this new series will contribute to fulfilling the program, which he initiated.

Jean Jacod (Paris)

Contents

Fractional Integrals and Extensions of Selfdecomposability

Ken-iti Sato

Abstract After characterizations of the class L of selfdecomposable distributions on $\mathbb{R}^d$ are recalled, the classes $K_{p,\alpha}$ and $L_{p,\alpha}$ with two continuous parameters $0 < p < \infty$ and $-\infty < \alpha < 2$ satisfying $K_{1,0} = L_{1,0} = L$ are introduced as extensions of the class L. They are defined as the classes of distributions of improper stochastic integrals $\int_0^{\infty-} f(s)dX_s^{(\rho)}$, where $f(s)$ is an appropriate non-random function and $X_s^{(\rho)}$ is a Lévy process on $\mathbb{R}^d$ with distribution ρ at time 1. The description of the classes is given by characterization of their Lévy measures, using the notion of monotonicity of order p based on fractional integrals of measures, and in some cases by addition of the condition of zero mean or some weaker conditions that are newly introduced – having weak mean 0 or having weak mean 0 absolutely. The class $L_{n,0}$ for a positive integer n is the class of n times selfdecomposable distributions. Relations among the classes are studied. The limiting classes as $p \to \infty$ are analyzed. The Thorin class T, the Goldie–Steutel–Bondesson class B, and the class L_∞ of completely selfdecomposable distributions, which is the closure (with respect to convolution and weak convergence) of the class $\mathfrak{S}$ of all stable distributions, appear in this context. Some subclasses of the class L_∞ also appear. The theory of fractional integrals of measures is built. Many open questions are mentioned.

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1 Introduction

1.1 Characterizations of Selfdecomposable Distributions

A distribution μ on $\mathbb{R}^d$ is called infinitely divisible if, for each positive integer n, there is a distribution μ_n such that

$$\mu = \underbrace{\mu_n * \mu_n * \cdots * \mu_n}_{n},$$

where $*$ denotes convolution. The class of infinitely divisible distributions on $\mathbb{R}^d$ is denoted by $ID = ID(\mathbb{R}^d)$. Let $C_\mu(z)$, $z \in \mathbb{R}^d$, denote the cumulant function of $\mu \in ID$, that is, the unique complex-valued continuous function on $\mathbb{R}^d$ with $C_\mu(0) = 0$ such that the characteristic function $\widehat{\mu}(z)$ of μ is expressed as $\widehat{\mu}(z) = e^{C_\mu(z)}$. If $\mu \in ID$, then $C_\mu(z)$ is expressed as

$$C_\mu(z) = -\frac{1}{2}\langle z, A_\mu z\rangle + \int_{\mathbb{R}^d} (e^{i\langle z,x\rangle} - 1 - i\langle z,x\rangle 1_{\{|x|\le 1\}}(x))\nu_\mu(dx) + i\langle \gamma_\mu, z\rangle. \quad (1.1)$$

Here $\langle z,x\rangle$ is the canonical inner product of z and x in $\mathbb{R}^d$, $|x| = \langle x,x\rangle^{1/2}$, $1_{\{|x|\le 1\}}$ is the indicator function of the set $\{|x| \le 1\}$, A_μ is a $d \times d$ symmetric nonnegative-definite matrix, called the Gaussian covariance matrix of μ, ν_μ is a measure on $\mathbb{R}^d$ satisfying $\nu_\mu(\{0\}) = 0$ and $\int_{\mathbb{R}^d}(|x|^2 \wedge 1)\nu_\mu(dx) < \infty$, called the Lévy measure of μ, and γ_μ is an element of $\mathbb{R}^d$. The triplet $(A_\mu, \nu_\mu, \gamma_\mu)$ is uniquely determined by μ. Conversely, to any triplet (A, ν, γ) there corresponds a unique $\mu \in ID$ such that $A = A_\mu$, $\nu = \nu_\mu$, and $\gamma = \gamma_\mu$. Throughout this article A_μ, ν_μ, and γ_μ are used in this sense.

A distribution μ on $\mathbb{R}^d$ is called *selfdecomposable* if, for any $b > 1$, there is a distribution μ_b such that

$$\widehat{\mu}(z) = \widehat{\mu}(b^{-1}z)\widehat{\mu}_b(z), \qquad z \in \mathbb{R}^d. \quad (1.2)$$

Let $L = L(\mathbb{R}^d)$ denote the class of selfdecomposable distributions on $\mathbb{R}^d$. It is characterized in the following four ways.

(a) A distribution μ on $\mathbb{R}^d$ is selfdecomposable if and only if $\mu \in ID$ and its Lévy measure ν_μ has a radial (or polar) decomposition

$$\nu_\mu(B) = \int_S \lambda(d\xi) \int_0^\infty 1_B(r\xi) r^{-1} k_\xi(r) dr \quad (1.3)$$

for Borel sets B in $\mathbb{R}^d$, where λ is a finite measure on the unit sphere $S = \{\xi \in \mathbb{R}^d \colon |\xi| = 1\}$ (if $d = 1$, then S is the two-point set $\{1, -1\}$) and $k_\xi(r)$ is a nonnegative function measurable in ξ and decreasing and right-continuous in r. (See Proposition 3.1 for an exact formulation of the radial decomposition.)

(b) Let $\{Z_k\colon k=1,2,\ldots\}$ be independent random variables on $\mathbb{R}^d$ and $Y_n = \sum_{k=1}^n Z_k$. Suppose that there are $b_n>0$ and $\gamma_n \in \mathbb{R}^d$ for $n=1,2,\ldots$ such that the law of $b_nY_n+\gamma_n$ converges weakly to a distribution μ as $n\to\infty$ and that $\{b_nZ_k\colon k=1,\ldots,n;\ n=1,2,\ldots\}$ is a null array (that is, for any $\varepsilon>0$, $\max_{1\le k\le n} P(|b_nZ_k|>\varepsilon)\to 0$ as $n\to\infty$). Then $\mu\in L$. Conversely, any $\mu\in L$ is obtained in this way.

(c) Given $\rho\in ID$, let $\{X_t^{(\rho)}: t\ge 0\}$ be a Lévy process on $\mathbb{R}^d$ (that is, a stochastic process continuous in probability, starting at 0, with time-homogeneous independent increments, with cadlag paths) having distribution ρ at time 1. If $\int_{|x|>1}\log|x|\rho(dx)<\infty$, then the improper stochastic integral $\int_0^{\infty-} e^{-s}dX_s^{(\rho)}$ is definable and its distribution

$$\mu = \mathscr{L}\left(\int_0^{\infty-} e^{-s}dX_s^{(\rho)}\right) \tag{1.4}$$

is selfdecomposable. Here $\mathscr{L}(Y)$ denotes the distribution (law) of a random element Y. Conversely, any $\mu\in L$ is obtained in this way. On the other hand, if $\int_{|x|>1}\log|x|\rho(dx)=\infty$, then $\int_0^{\infty-} e^{-s}dX_s^{(\rho)}$ is not definable. (See Section 3.4 for improper stochastic integrals.)

To see that μ of (1.4) is selfdecomposable, notice that

$$\int_0^{\infty-} e^{-s}dX_s^{(\rho)} = \int_0^{\log b} e^{-s}dX_s^{(\rho)} + \int_{\log b}^{\infty-} e^{-s}dX_s^{(\rho)} = I_1+I_2,$$

I_1 and I_2 are independent, and

$$I_2 = \int_0^{\infty-} e^{-\log b - s}dX_{\log b+s}^{(\rho)} = b^{-1}\int_0^{\infty-} e^{-s}dY_s^{(\rho)},$$

where $\{Y_s^{(\rho)}\}$ is identical in law with $\{X_s^{(\rho)}\}$, and hence μ satisfies (1.2).

(d) Let $\{Y_t: t\ge 0\}$ be an additive process on $\mathbb{R}^d$, that is, a stochastic process continuous in probability with independent increments, with cadlag paths, and with $Y_0=0$. If, for some $H>0$, it is H-selfsimilar (that is, for any $a>0$, the two processes $\{Y_{at}: t\ge 0\}$ and $\{a^HY_t: t\ge 0\}$ have an identical law), then the distribution μ of Y_1 is in L. Conversely, for any $\mu\in L$ and $H>0$, there is a process $\{Y_t: t\ge 0\}$ satisfying these conditions and $\mathscr{L}(Y_1)=\mu$.

Historically, selfdecomposable distributions were introduced by Lévy [18] in 1936 and written in his 1937 book [19] under the name "lois-limites", to characterize the limit distributions in (b). Lévy wrote in [18, 19] that this characterization problem had been posed by Khintchine, and Khintchine's book [16] in 1938 called these distributions "of class L". The book [9] of Gnedenko and Kolmogorov uses the same naming. Loève's book [20] uses the name "selfdecomposable".

The property (c) gives a characterization of the stationary distribution of an Ornstein–Uhlenbeck type process (sometimes called an Ornstein–Uhlenbeck process driven by a Lévy process) $\{V_t: t\ge 0\}$ defined by

$$V_t = e^{-t}V_0 + \int_0^t e^{s-t}dX_s^{(\rho)},$$

where V_0 and $\{X_t^{(\rho)}: t \geq 0\}$ are independent. The stationary Ornstein–Uhlenbeck type process and the selfsimilar process in the property (d) are connected via the so-called Lamperti transformation (see [11, 26]). For historical facts concerning (c) see [33], pp. 54–55.

The proofs of (a)–(d) and many examples of selfdecomposable distributions are found in Sato's book [39].

The main purpose of the present article is to give two families of subclasses of ID, with two continuous parameters, related to L, using improper stochastic integrals and extending the characterization (c) of L.

1.2 Nested Classes of Multiply Selfdecomposable Distributions

If $\mu \in L$, then, for any $b > 1$, the distribution μ_b in (1.2) is infinitely divisible and uniquely determined by μ and b. If $\mu \in L$ and $\mu_b \in L$ for all $b > 1$, then μ is called *twice selfdecomposable*. Let n be a positive integer ≥ 3. A distribution μ is called *n times selfdecomposable*, if $\mu \in L$ and if μ_b is $n-1$ times selfdecomposable. Let $L_{1,0} = L_{1,0}(\mathbb{R}^d) = L(\mathbb{R}^d)$ and let $L_{n,0} = L_{n,0}(\mathbb{R}^d)$ be the class of n times selfdecomposable distributions on $\mathbb{R}^d$. Then we have

$$ID \supset L = L_{1,0} \supset L_{2,0} \supset L_{3,0} \supset \cdots . \tag{1.5}$$

These classes and the class $L_\infty(\mathbb{R}^d)$ in Section 1.4 were introduced by Urbanik [52, 53] and studied by Sato [37] and others. (In [37, 52, 53] the class $L_{n,0}$ is written as L_{n-1}, but this notation is inconvenient in this article.)

An n times selfdecomposable distribution is characterized by the property that $\mu \in ID$ with Lévy measure ν_μ having radial decomposition (1.3) in (a) with $k_\xi(r) = h_\xi(\log r)$ for some function $h_\xi(y)$ monotone of order n for each ξ (see Section 1.5 and Proposition 2.11 for the monotonicity of order n). In property (b), $\mu \in L_{n,0}$ is characterized by the property that $\mathscr{L}(Z_k) \in L_{n-1,0}$ for $k = 1, 2, \ldots$. In (c), $\mu \in L_{n,0}$ is characterized by $\rho \in L_{n-1,0}$ in (1.4). A direct generalization of (1.4) using $\exp(-s^{1/n})$ or, equivalently, $\exp(-(n!\,s)^{1/n})$ in place of e^{-s} is also possible. In (d), $\mu \in L_{n,0}$ if and only if, for any H, the corresponding process $\{Y_t: t \geq 0\}$ satisfies $\mathscr{L}(Y_t - Y_s) \in L_{n-1,0}$ for $0 < s < t$. The proofs are given in [12, 25, 33, 37].

1.3 Continuous-Parameter Extension of Multiple Selfdecomposability

In 1980s Nguyen Van Thu [49–51] defined a continuous-parameter extension of $L_{n,0}$, replacing the positive integer n by a real number $p > 0$. He introduced fractional times multiple Selfdecomposability and used fractional integrals and fractional difference quotients. On the one hand he extended the definition of n times

Selfdecomposability based on (1.2) to fractional times Selfdecomposability in the form of infinite products. On the other hand he extended essentially the formula (1.4) in the characterization (c), considering its Lévy measure.

Directly using improper stochastic integrals with respect to Lévy processes, we will define and study the decreasing classes $L_{p,0}$ for $p > 0$, which generalize the nested classes $L_{n,0}$ for $n = 1, 2, \ldots$. Thus the results of Thu will be reformulated as a special case in a family $L_{p,\alpha}$ with two continuous parameters $0 < p < \infty$ and $-\infty < \alpha < 2$. The definition of $L_{p,\alpha}$ will be given in Section 1.6.

1.4 Stable Distributions and the Class L_∞

Let μ be a distribution on $\mathbb{R}^d$. Let $0 < \alpha \le 2$. We say that μ is *strictly α-stable* if $\mu \in ID$ and, for any $t > 0$, $\widehat{\mu}(z)^t = \widehat{\mu}(t^{1/\alpha} z)$, $z \in \mathbb{R}^d$. We say that μ is *α-stable* if $\mu \in ID$ and, for any $t > 0$, there is $\gamma_t \in \mathbb{R}^d$ such that $\widehat{\mu}(z)^t = \widehat{\mu}(t^{1/\alpha} z) \exp(i\langle \gamma_t, z\rangle)$, $z \in \mathbb{R}^d$. (When μ is a δ-distribution, this terminology is not the same as in Sato [39].) Let $\mathfrak{S}_\alpha^0 = \mathfrak{S}_\alpha^0(\mathbb{R}^d)$ and $\mathfrak{S}_\alpha = \mathfrak{S}_\alpha(\mathbb{R}^d)$ be the class of strictly α-stable distributions on $\mathbb{R}^d$ and the class of α-stable distributions on $\mathbb{R}^d$, respectively. Let $\mathfrak{S} = \mathfrak{S}(\mathbb{R}^d)$ be the class of stable distributions on $\mathbb{R}^d$. That is, $\mathfrak{S} = \bigcup_{0<\alpha\le 2} \mathfrak{S}_\alpha$. A distribution $\mu \in ID$ is in $\mathfrak{S}_2$ if and only if $\nu_\mu = 0$, that is, μ is Gaussian. A distribution $\mu \in ID$ is in $\mathfrak{S}_\alpha$ with $0 < \alpha < 2$ if and only if $A_\mu = 0$ and ν_μ has a radial decomposition (1.3) with $k_\xi(r) = r^{-\alpha}$. A distribution $\mu \in \mathfrak{S}_\alpha$ with $1 < \alpha \le 2$ is in $\mathfrak{S}_\alpha^0$ if and only if μ has mean 0. A distribution $\mu \in \mathfrak{S}_1$ is in $\mathfrak{S}_1^0$ if and only if ν_μ has a radial decomposition (1.3) with $k_\xi(r) = r^{-1}$ and $\int_S \xi \lambda(d\xi) = 0$. A distribution $\mu \in \mathfrak{S}_\alpha$ with $0 < \alpha < 1$ is in $\mathfrak{S}_\alpha^0$ if and only if it is driftless in the sense that

$$C_\mu(z) = \int_S \lambda(d\xi) \int_0^\infty (e^{i\langle r\xi, z\rangle} - 1) r^{-\alpha-1} dr, \qquad z \in \mathbb{R}^d.$$

Lots of results are accumulated on stable distributions and processes. To mention one of them, the asymptotic behavior of the density of $\mu \in \mathfrak{S}_\alpha(\mathbb{R}^d)$, $d \ge 2$, $\alpha \in (0,2)$, sensitively depends on the radial direction and exhibits amazing diversity, as Watanabe [54] shows.

Let $L_\infty = L_\infty(\mathbb{R}^d)$ denote the smallest class that is closed under convolution and weak convergence and contains $\mathfrak{S}(\mathbb{R}^d)$. This class was introduced by Urbanik [52, 53] and reformulated by Sato [37]. If $\mu \in L_\infty$, then $\mu \in ID$ with Lévy measure ν_μ being such that

$$\nu_\mu(B) = \int_{(0,2)} \Gamma(d\alpha) \int_S \lambda_\alpha(d\xi) \int_0^\infty 1_B(r\xi) r^{-\alpha-1} dr \tag{1.6}$$

for Borel sets B in $\mathbb{R}^d$, where Γ is a measure on the open interval $(0,2)$ satisfying

$$\int_{(0,2)} (\alpha^{-1} + (2-\alpha)^{-1}) \Gamma(d\alpha) < \infty \tag{1.7}$$

and $\{\lambda_\alpha\colon \alpha \in (0,2)\}$ is a measurable family of probability measures on S. This Γ is determined by ν_μ, and λ_α is determined by ν_μ up to α of Γ-measure 0. Conversely, if a measure ν on $\mathbb{R}^d$ is expressed by the right-hand side of (1.6) with some Γ and λ_α satisfying the conditions above, then, for any A and γ, (A,ν,γ) is the triplet of some $\mu \in L_\infty$.

We will also use the class $L_\infty^E = L_\infty^E(\mathbb{R}^d)$ for a Borel subset E of the open interval $(0,2)$; this is the class of $\mu \in L_\infty$ whose measure Γ is concentrated on E.

Another characterization of $L_\infty(\mathbb{R}^d)$ is that $\mu \in L_\infty$ if and only if $\mu \in L$ and ν_μ has a radial decomposition (1.3) with $k_\xi(r) = h_\xi(\log r)$ where h_ξ is a completely monotone function on $\mathbb{R}$ for each ξ. Hence we have

$$L_\infty = \bigcap_{n=1,2,\ldots} L_{n,0}. \tag{1.8}$$

Thus distributions in L_∞ are sometimes called *completely selfdecomposable*.

Zinger [57] introduced a subclass $\mathscr{P}_r$ (r being a positive integer) of the class $L(\mathbb{R})$; it is defined to be the class of limit distributions μ in (b) of Section 1.1 such that $\{\mathscr{L}(Z_k)\colon k = 1,2,\ldots\}$ consists of at most r different distributions on $\mathbb{R}$. It is known that $\mathscr{P}_1 = \mathfrak{S}(\mathbb{R})$ and that $\mu \in \mathscr{P}_2$ if and only if μ is the convolution of at most two stable distributions. In [57] a beautiful explicit description of the Lévy measures of distributions in $\mathscr{P}_r$ is given and it is shown that a distribution in $\mathscr{P}_r$ with $r \geq 3$ is not necessarily the convolution of stable distributions on $\mathbb{R}$. Any μ in $\mathscr{P}_r$ is the convolution of at most r semi-stable distributions of a special form. However, no other characterization of $\mathscr{P}_r$ exists, as far as the author knows.

1.5 Fractional Integrals

The key concept to connect the representation of Lévy measures for the class $L(\mathbb{R}^d)$ and that for the class $L_\infty(\mathbb{R}^d)$ is monotonicity of order $p \in (0,\infty)$. It is defined by using the notion of fractional integrals or Riemann–Liouville integrals. Let us write

$$\Gamma_p = \Gamma(p), \quad c_p = 1/\Gamma(p)$$

throughout this article. The fractional integral of order $p > 0$ of a function $f(s)$ on $\mathbb{R}$ in a suitable class is given by

$$c_p \int_r^\infty (s-r)^{p-1} f(s)ds,$$

which is the interpolation ($1 \leq p < \infty$) and extrapolation ($0 < p \leq 1$) of the n times integration

$$\int_r^\infty ds_n \int_{s_n}^\infty ds_{n-1} \cdots \int_{s_2}^\infty f(s_1)ds_1 = \frac{1}{(n-1)!}\int_r^\infty (s-r)^{n-1} f(s)ds.$$