Graduate Texts in Mathematics 99

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Finite Reflection Groups

Second Edition

With 38 Illustrations

Springer-Verlag
New York Berlin Heidelberg Tokyo
World Publishing Corporation, Beijing, China

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AMS Subject Classifications: 20-01, 20B25, 51F15, 20G45, 20H15

Library of Congress Cataloging in Publication Data
Grove, Larry C.
Finite reflection groups.
(Graduate texts in mathematics ; 99)
Rev. ed. of: Finite reflection groups / C. T. Benson, L. C. Grove. 1971.
Bibliography: p.
Includes index.
1. Finite groups. 2. Transformations (Mathematics)
I. Benson, C. T. (Clark T.) II. Benson, C. T. (Clark T.) Finite reflection groups. III. Title. IV. Series.
QA171.G775 1985 512'.2 84-20234

The first edition of this book was published by Bogden & Quigley, Inc.: New York and California, in 1971.

Printed and bound by R. R. Donnelley & Sons, Harrisonburg, Virginia.

Reprinted in China by World Publishing Corporation
For distribution and sale in the People's Republic of China only
只限在中华人民共和国发行

ISBN 0-387-96082-1 Springer-Verlag New York Berlin Heidelberg Tokyo
ISBN 3-540-96082-1 Springer-Verlag Berlin Heidelberg New York Tokyo
ISBN 7-5062-0128-3 World Publishing Corporation China

PREFACE TO THE SECOND EDITION

This edition differs from the original mainly by the addition of a seventh chapter, on the classical invariant theory of finite reflection groups. Most of the changes in the original six chapters are corrections of misprints and minor errors. We are indebted, however, to Klaus Benkert of the RWTH Aachen for pointing out to us Proposition 5.1.5, making possible a neater discussion of the positive definiteness of marked graphs. We have also added an appendix listing the Schoenflies and International notations for crystallographic point groups.

Since many beginning German courses in the United States seem no longer to include an introduction to German script, it may be helpful to some readers if the script letters used in Chapter 7 are introduced here with their Roman counterparts.

German Script	$\mathfrak{a}$	$\mathfrak{b}$	$\mathfrak{c}$	$\mathfrak{F}$	$\mathfrak{I}$	$\mathfrak{K}$	$\mathfrak{L}$	$\mathfrak{P}$	$\mathfrak{Q}$
Roman	a	b	c	F	I	K	L	P	Q

Our thanks go to David Surowski and Dick Pierce for reading drafts of Chapter 7 and suggesting corrections and improvements, to Helen Grove for typing the new chapter and, belatedly, to Sandra Grove for proofreading the first six.

August 1984 L.C.G. AND C.T.B.

PREFACE TO THE FIRST EDITION

This book began as lecture notes for a course given at the University of Oregon. The course, given for undergraduates and beginning graduate students, follows immediately after a conventional course in linear algebra and serves two chief pedagogical purposes. First, it reinforces the students' newly won knowledge of linear algebra by giving applications of several of the theorems they have learned and by giving geometrical interpretations for some of the notions of linear algebra. Second, some students take the course before or concurrently with abstract algebra, and they are armed in advance with a collection of fairly concrete nontrivial examples of groups.

The first comprehensive treatment of finite reflection groups was given by H. S. M. Coxeter in 1934. In [9] he completely classified the groups and derived several of their properties, using mainly geometrical methods. He later included a discussion of the groups in his book *Regular Polytopes* [10]. Another discussion, somewhat more algebraic in nature, was given by E. Witt in 1941 [37]. An algebraic account of reflection groups was presented by P. Cartier in the Chevalley Seminar reports (see [6]). Another has recently appeared in N. Bourbaki's chapters on Lie groups and Lie algebras [3].

Since the sources cited above do not seem to be easily accessible to most undergraduates, we have attempted to give a discussion of finite reflection groups that is as elementary as possible. We have tried to reach a middle ground between Coxeter and Bourbaki. Our approach is algebraic, but we have retained some of the geometrical flavor of Coxeter's approach.

Chapter 1 introduces some of the terminology and notation used later and indicates prerequisites. Chapter 2 gives a reasonably thorough account of *all* finite subgroups of the orthogonal groups in two and three dimensions. The presentation is somewhat less formal than in succeeding chapters. For instance, the existence of the icosahedron is accepted as an empirical fact, and no formal proof of existence is included. Throughout most of Chapter 2 we do not distinguish between groups that are "geometrically indistinguishable," that is, conjugate in the orthogonal group. Very little of the material in Chapter 2 is actually required for the subsequent chapters, but it serves two important purposes: It aids in the development of geometrical insight, and it serves as a source of illustrative examples.

There is a discussion of fundamental regions in Chapter 3. Chapter 4 provides a correspondence between fundamental reflections and fundamental regions via a discussion of root systems. The actual classification and construction of finite reflection groups takes place in Chapter 5, where we have in part followed the methods of E. Witt and B. L. van der Waerden. Generators and relations for finite reflection groups are discussed in Chapter 6. There are historical remarks and suggestions for further reading in a Postlude.

Since we have written with the student in mind we have included considerable detail and a number of illustrative examples. Exercises are included in every chapter but the first. The results of some of the exercises are used in the body of the text. The list of identifications in Exercise 5.7 was worked out by one of our students, Leslie Wilson.

We wish to thank James Humphreys, Otto Kegel, and Louis Solomon for reading the manuscript and making numerous excellent suggestions. We also derived considerable benefit from Charles Curtis's lectures on root systems and Chevalley groups.

July 1970 C.T.B. AND L.C.G.

CONTENTS

chapter 1

PRELIMINARIES

1.1 LINEAR ALGEBRA

We assume that the reader is familiar with the contents of a standard course in linear algebra, including finite-dimensional vector spaces, subspaces, linear transformations and matrices, determinants, eigenvalues, bilinear and quadratic forms, positive definiteness, inner product spaces, and orthogonal linear transformations. Accounts of these topics may be found in most linear algebra books (e.g., [14] or [21]). Throughout the book V will denote a real Euclidean vector space, i.e., a finite-dimensional inner product space over the real field $\mathcal{R}$. Partly in order to establish notation we list some of the properties of V that are of importance for the ensuing discussion.

If X and Y are subsets of V such that $(x, y) = 0$ for all $x \in X$ and all $y \in Y$, we shall say that X and Y are *orthogonal*, or *perpendicular*, and write $X \perp Y$. If $X \subseteq V$, the *orthogonal complement* of X, which is the subspace of V consisting of all $x \in V$ such that $x \perp X$, will be denoted by $X^{\perp}$. If W is a subspace of V, then $W^{\perp\perp} = W$ and $V = W \oplus W^{\perp}$.

If $\{x_1, \ldots, x_n\}$ is a basis for V, let V_i be the subspace spanned by $\{x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n\}$, excluding x_i. If $0 \neq y_i \in V_i^{\perp}$, then $(x_j, y_i) = 0$ for all $j \neq i$, but $(x_i, y_i) \neq 0$, for otherwise $y_i \in V^{\perp} = 0$. Dividing y_i by (x_i, y_i), if necessary, we may assume that $(x_i, y_i) = 1$, thereby making y_i unique since $\dim(V_i^{\perp}) = 1$. Observe that if $\Sigma_{i=1}^{n} \lambda_i y_i = 0$ with $\lambda_i \in \mathcal{R}$, then

$$0 = (x_j, 0) = (x_j, \Sigma_i \lambda_i y_i) = \Sigma_i \lambda_i (x_j, y_i) = \lambda_j$$

for all j, and so $\{y_1, \ldots, y_n\}$ is linearly independent. Thus $\{y_1, \ldots, y_n\}$ is a basis, called the *dual basis* of $\{x_1, \ldots, x_n\}$. It is the unique basis with the property that

$$(x_i, y_j) = \delta_{ij} = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases}$$

The space of all n-tuples (column vectors) of real numbers will be denoted by $\mathscr{R}^n$. Since there is seldom any chance of confusion we shall often write the elements of $\mathscr{R}^n$ as row vectors $(\lambda_1, \ldots, \lambda_n)$ for the sake of typographical convenience. The usual basis vectors along the positive coordinate axes in $\mathscr{R}^n$ will be denoted by

$$e_1 = (1, 0, \ldots, 0), \qquad e_2 = (0, 1, 0, \ldots, 0),$$

etc. The space $\mathscr{R}^n$ is an inner product space, with

$$((\lambda_1, \ldots, \lambda_n), (\mu_1, \ldots, \mu_n)) = \Sigma_{i=1}^n \lambda_i \mu_i.$$

If V is any real Euclidean vector space, then it is a consequence of the Gram–Schmidt theorem ([14], p. 108) that there is an inner product preserving isomorphism from V onto $\mathscr{R}^n$, where $n = \dim V$. Thus when it is convenient we shall lose no generality if we assume that $V = \mathscr{R}^n$.

The *length* $\sqrt{(x, x)}$ of a vector $x \in V$ will be denoted by $\|x\|$. If $x, y \in V$, then the *distance* between them, denoted by $d(x, y)$, is defined to be $\|x - y\|$. For a fixed vector $x_0 \in V$ and real number $\varepsilon > 0$ the set

$$\{x \in V : d(x, x_0) = \varepsilon\}$$

is called the *sphere* of radius ε centered at x_0, and the set

$$\{x \in V : d(x, x_0) < \varepsilon\}$$

is called the (open) *ball* of radius ε centered at x_0.

A subset U of V is called *open* if and only if given any $x \in U$ there is some $\varepsilon > 0$ for which the ball of radius ε centered at x lies entirely within U. The conditions of the definition are vacuously satisfied by the empty set $\varnothing$, so $\varnothing$ is open by default. Note that finite intersections and arbitrary unions of open sets are open. A subset D of V is called *closed* if and only if its complement $V \setminus D$ is open, so finite unions and arbitrary intersections of closed sets are closed. The intersection of all closed sets containing a set X is called the *closure* of X and is denoted by X^-. The *interior* X^0 of X is the union of all open subsets of X. The *boundary* of X is defined to be $X^- \setminus X^0$. For example, the sphere of radius ε centered at x_0 is the boundary of the ball with the same radius and center.

If X is a fixed subset of V and $Y \subseteq X$, then Y is called *relatively open* in X if and only if $Y = X \cap U$ for some open subset U of V. Likewise,

Y is *relatively closed* in X if and only if $Y = X \cap D$ for some closed subset D of V, and the (*relative*) *closure* of Y in X is the intersection of X with the closure Y^{-} of Y in V. A subset X of V is *connected* if and only if it is not the disjoint union of two nonempty relatively open subsets. At the opposite extreme X is *discrete* if and only if every point of X is a relatively open set.

If $\dim V = n$, then a *hyperplane* in V is an $(n-1)$-dimensional subspace. A *line* in V is any translate of a one-dimensional subspace. Thus a line is a subset of the form $\{x + \lambda y : \lambda \in \mathscr{R}\}$, where x and y are fixed vectors with $y \neq 0$. The *line segment* $[xy]$ between two vectors x and y of V is the set

$$\{x + \lambda(y - x) : 0 \leq \lambda \leq 1\}.$$

Note that if $x \neq y$, then $[xy]$ is the smallest connected subset of the line

$$\{x + \lambda(y - x) : \lambda \in \mathscr{R}\}$$

that contains x and y. A subset X of V is called *convex* if and only if the line segment $[xy]$ lies wholly within X for all points x and y of X. Observe that a convex set is connected.

A *transformation* of V is understood to be a linear transformation. The group of all orthogonal transformations of V will be denoted by $\mathscr{O}(V)$. If $T \in \mathscr{O}(V)$ then $\det T = \pm 1$, and if a (complex) number λ is an eigenvalue of T then $|\lambda| = 1$. If $T \in \mathscr{O}(V)$ and $\det T = 1$, then T will be called a *rotation*.

The ring of integers will be denoted by $\mathbb{Z}$.

1.2 GROUP THEORY

We shall assume that the reader is familiar with the following notions from elementary group theory: subgroup, coset, order, index, homomorphism, kernel, normal subgroup, isomorphism, and direct product. A discussion may be found in any book on group theory or almost any book on abstract algebra (e.g., [20], [1], or [23]).

If $\mathscr{S}$ is a set, the cardinality of $\mathscr{S}$ will be denoted by $|\mathscr{S}|$. In particular, the order of a group $\mathscr{G}$ is $|\mathscr{G}|$. If $\mathscr{S}$ is a subset of a group $\mathscr{G}$, then $\langle \mathscr{S} \rangle$ will denote the subgroup of $\mathscr{G}$ generated by $\mathscr{S}$. If $\mathscr{H}$ is a subgroup of $\mathscr{G}$ we write $\mathscr{H} \leq \mathscr{G}$, and $[\mathscr{G} : \mathscr{H}]$ will denote the index of $\mathscr{H}$ in $\mathscr{G}$.

A *permutation* of a set $\mathscr{S}$ is a 1-1 function from $\mathscr{S}$ onto $\mathscr{S}$. The set $\mathscr{P}(\mathscr{S})$ of all permutations of $\mathscr{S}$ is a group under the operation of composition of functions; i.e., $(fg)(x) = f(g(x))$, all $x \in \mathscr{S}$. If $\mathscr{S} = \{1, 2, \ldots, n\}$, then the group $\mathscr{P}(\mathscr{S})$ is called the *symmetric group* on n letters and is denoted by $\mathscr{S}_n$. We shall assume known the elementary properties of $\mathscr{S}_n$

(see [23], pp. 64–68). In particular, $\mathscr{S}_n$ has a subgroup of index 2, the *alternating group* on n letters, consisting of all the even permutations in $\mathscr{S}_n$.

If $\mathscr{S}$ is a set, then a group $\mathscr{G}$ is said to be (represented as) a *permutation group* on $\mathscr{S}$ if and only if there is a homomorphism φ from $\mathscr{G}$ to $\mathscr{P}(\mathscr{S})$. If φ is an isomorphism into $\mathscr{P}(\mathscr{S})$, then $\mathscr{G}$ is said to be represented *faithfully* or to be a *faithful* permutation group on $\mathscr{S}$. Note that if $\mathscr{G}$ is faithful and $\mathscr{S}$ is finite, then $\mathscr{G}$ is isomorphic with a subgroup of $\mathscr{S}_n$, and in particular $\mathscr{G}$ is finite.

If $\mathscr{G}$ is a permutation group on $\mathscr{S}$, we shall write simply Tx rather than $(\varphi T)x$ for all $T \in \mathscr{G}$, $x \in \mathscr{S}$. If $x \in \mathscr{S}$, then the subset $\mathscr{H}$ of $\mathscr{G}$, consisting of all $T \in \mathscr{G}$ for which $Tx = x$, is a subgroup called the *stabilizer* of x, denoted by Stab(x). The subset of $\mathscr{S}$ consisting of all Tx, as T ranges over $\mathscr{G}$, is called the *orbit* of x, denoted by Orb(x). If Orb(x) $= \mathscr{S}$ for each $x \in \mathscr{S}$, then $\mathscr{G}$ is said to be *transitive* on $\mathscr{S}$.

Proposition 1.2.1

If $\mathscr{G}$ is a permutation group on a set $\mathscr{S}$ and $x \in \mathscr{S}$, then $[\mathscr{G} : \mathrm{Stab}(x)] = |\mathrm{Orb}(x)|$.

Proof

Set $\mathscr{H} = \mathrm{Stab}(x)$. If $R, T \in \mathscr{G}$ and $R\mathscr{H} = T\mathscr{H}$, then $T^{-1}R \in \mathscr{H}$, or $T^{-1}Rx = x$; so $Rx = Tx$. Thus $\theta(T\mathscr{H}) = Tx$ defines a mapping θ from the set of left cosets of $\mathscr{H}$ onto the orbit of x. If $Rx = Tx$, then $T^{-1}R \in \mathscr{H}$, and $R\mathscr{H} = T\mathscr{H}$. Thus θ is also 1-1 and the proposition is proved.

chapter 2

FINITE GROUPS IN TWO AND THREE DIMENSIONS

2.1 *ORTHOGONAL TRANSFORMATIONS IN TWO DIMENSIONS*

If $T \in \mathcal{O}(\mathcal{R}^2)$, then T is completely determined by its action on the basis vectors $e_1 = (1, 0)$ and $e_2 = (0, 1)$. If $Te_1 = (\mu, \nu)$, then $\mu^2 + \nu^2 = 1$ and $Te_2 = \pm(-\nu, \mu)$, since T preserves length and orthogonality. Choose $\theta, 0 \leq \theta < 2\pi$, such that $\cos\theta = \mu$ and $\sin\theta = \nu$.

If $Te_2 = (-\nu, \mu)$, then T is represented by the matrix

$$A = \begin{bmatrix} \mu & -\nu \\ \nu & \mu \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix},$$

and it is clear that T is a counterclockwise rotation of the plane about the origin through the angle θ (see Figure 2.1). Observe that

$$\det T = \mu^2 + \nu^2 = \cos^2\theta + \sin^2\theta = 1.$$

If $Te_2 = (\nu, -\mu)$, then T is represented by the matrix

$$B = \begin{bmatrix} \mu & \nu \\ \nu & -\mu \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix}.$$

In this case observe that

$$\det T = -\cos^2\theta - \sin^2\theta = -1,$$

and that

$$B^2 = \begin{bmatrix} \mu^2 + \nu^2 & 0 \\ 0 & \mu^2 + \nu^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

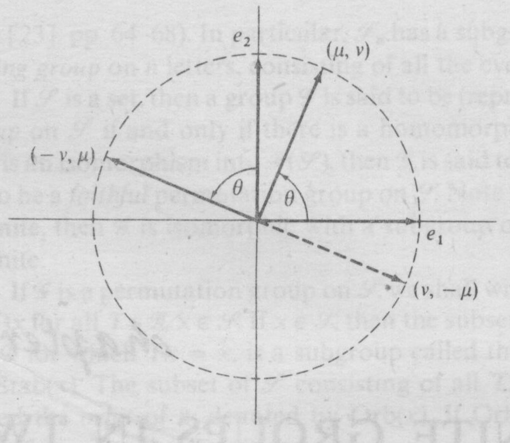

Figure 2.1

so that $T^2 = 1$. It is easy to verify (Exercise 2.1) that the vector $x_1 = (\cos\theta/2, \sin\theta/2)$ is an eigenvector having eigenvalue 1 for T, so that the line $l = \{\lambda x_1 : \lambda \in \mathscr{R}\}$ is left pointwise fixed by T. Similarly, the vector $x_2 = (-\sin\theta/2, \cos\theta/2)$ is an eigenvector with eigenvalue -1, and $x_2 \perp x_1$ [see Figure 2.2(a)]. With respect to the basis $\{x_1, x_2\}$ the transformation T is represented by the matrix

$$C = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

If $x = \lambda_1 x_1 + \lambda_2 x_2$, then $Tx = \lambda_1 x_1 - \lambda_2 x_2$, and T sends x to its mirror

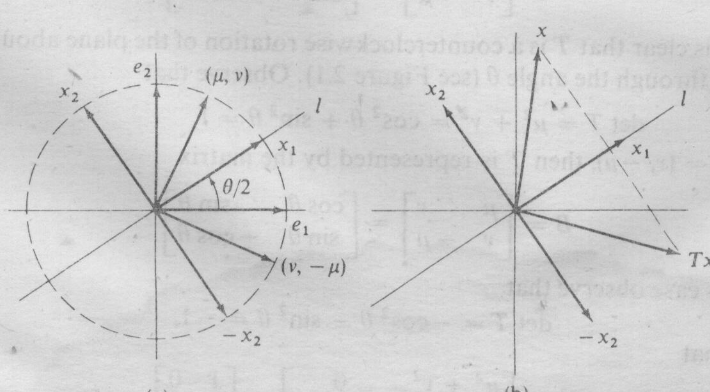

Figure 2.2

image with respect to the line l [see Figure 2.2(b)]. The transformation T is called the *reflection through* l or the *reflection along* x_2. Observe that

$$Tx = x - 2(x, x_2)x_2$$

for all $x \in \mathscr{R}^2$.

We have shown that every orthogonal transformation of $\mathscr{R}^2$ is either a rotation or a reflection.

2.2 *FINITE GROUPS IN TWO DIMENSIONS*

Suppose that $\dim V = 2$ and that $\mathscr{G}$ is a finite subgroup of $\mathscr{O}(V)$. The set of all rotations in $\mathscr{G}$ constitutes a subgroup $\mathscr{H}$ of $\mathscr{G}$. As was shown in Section 2.1, each $T \in \mathscr{H}$ is a counterclockwise rotation of V through an angle $\theta = \theta(T)$ with $0 \leq \theta < 2\pi$. If $\mathscr{H} \neq 1$, choose $R \in \mathscr{H}$ with $R \neq 1$, for which $\theta(R)$ is minimal. If $T \in \mathscr{H}$, choose an integer m such that

$$m\theta(R) \leq \theta(T) < (m+1)\theta(R).$$

Then $0 \leq \theta(T) - m\theta(R) < \theta(R)$. But

$$\theta(T) - m\theta(R) = \theta(R^{-m}T),$$

since $R^{-m}T$ is a counterclockwise rotation through angle $\theta(T)$ followed by m *clockwise* rotations, each through angle $\theta(R)$. Since $\theta(R)$ was chosen to be minimal, we must have $\theta(R^{-m}T) = 0$; so $R^{-m}T = 1$ or $T = R^m$. In other words, $\mathscr{H} = \langle R \rangle$ is a cyclic group. It also follows that $\theta(R) = 2\pi/n$, where $n = |\mathscr{H}|$.

If $\mathscr{G} = \mathscr{H}$, we have shown that $\mathscr{G}$ is a cyclic group of order n, in which case $\mathscr{G}$ will be denoted by $\mathscr{C}_2^n$ (the subscript calls attention to the fact that $\dim V = 2$).

Suppose next that $\mathscr{G} \neq \mathscr{H}$, and choose a reflection $S \in \mathscr{G}$. Since $\det(SR^k) = \det S = -1$ for all integers k, the coset $S\mathscr{H}$ contains $n = |\mathscr{H}|$ distinct reflections. If $T \in \mathscr{G}$ is a reflection, then

$$\det(ST) = (\det S)(\det T) = (-1)(-1) = 1,$$

so $ST \in \mathscr{H}$; hence $T \in S\mathscr{H}$, since $S^{-1} = S$. Thus $\mathscr{H}$ is a subgroup of index 2 in $\mathscr{G}$, and if $\mathscr{H} = \langle R \rangle$, as above, then

$$\mathscr{G} = \langle R, S \rangle = \{1, R, \ldots, R^{n-1}, S, SR, \ldots, SR^{n-1}\},$$

and $|\mathscr{G}| = 2n$. Since RS is a reflection, we have $(RS)^2 = 1$, or $RS = SR^{-1} = SR^{n-1}$, completely determining the multiplication in $\mathscr{G}$. The group $\mathscr{G}$ is called the *dihedral group* of order $2n$, and it will be denoted by $\mathscr{H}_2^n$. We have proved

Theorem 2.2.1

If $\dim V = 2$ and $\mathscr{G}$ is a finite subgroup of $\mathscr{O}(V)$, then $\mathscr{G}$ is either a cyclic group $\mathscr{C}_2^n$ or a dihedral group $\mathscr{H}_2^n$, $n = 1, 2, 3, \ldots$.

If we set $T = RS$ in the dihedral group $\mathscr{H}_2^n = \langle S, R\rangle$, then T is a reflection, since $\det T = -1$. Since $TS = RS^2 = R$, it is clear that $\langle S, T\rangle = \mathscr{H}_2^n$, so $\mathscr{H}_2^n$ is generated by reflections. If we suppose that the orthonormal basis $\{x_1, x_2\}$ of eigenvectors of S discussed in Section 2.1 coincides with the usual basis $\{e_1, e_2\}$ in $\mathscr{R}^2$, then we may assume that S and R are represented by the matrices

$$A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} \cos 2\pi/n & -\sin 2\pi/n \\ \sin 2\pi/n & \cos 2\pi/n \end{bmatrix},$$

respectively. Thus T is represented by the matrix

$$C = BA = \begin{bmatrix} \cos 2\pi/n & \sin 2\pi/n \\ \sin 2\pi/n & -\cos 2\pi/n \end{bmatrix},$$

so T is a reflection through a line l inclined at an angle of π/n to the positive x-axis. Let us use these ideas to give a geometrical interpretation of the group $\mathscr{H}_2^n$.

Denote by F the open wedge-shaped region in the first quadrant bounded by the x-axis and the line l. The x-axis is a reflecting line for the transformation S, and l is a reflecting line for the transformation T. The $2n$ congruent regions in the plane obtained by rotating the region F through successive multiples of π/n can be labeled with the elements of $\mathscr{H}_2^n$ as follows: For each $U \in \mathscr{H}_2^n$, designate by U the region $U(F)$ obtained by applying U to all points of the region F.

The procedure is illustrated in Figure 2.3 for the case $n = 4$. If two plane mirrors are set facing one another along the reflecting lines for S

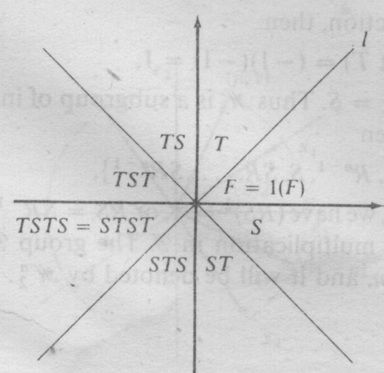

Figure 2.3

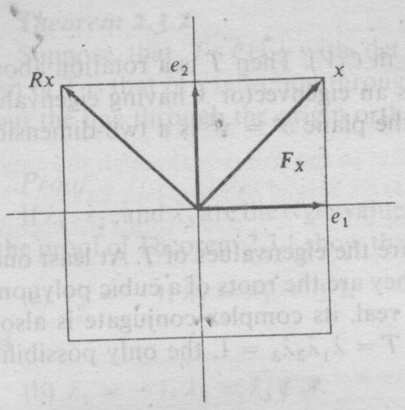

Figure 2.4

and T, with their common edge perpendicular to the plane at the origin, then the other lines may be seen in the mirrors as edges of virtual mirrors. If an object is placed between the mirrors in the region F, then reflections of the object can be seen in the seven images of F. This illustrates the principle of the kaleidoscope and shows a connection between the kaleidoscope and the dihedral groups.

Observe that the region F is open, that no point of F is mapped to any other point of F by any nonidentity element U of $\mathscr{H}_2^n$, and that the union of the closures $(UF)^-$, $U \in \mathscr{H}_2^n$, is all of $\mathscr{R}^2$. A region F with these properties will be called a *fundamental region* for the group $\mathscr{H}_2^n$. Fundamental regions will be discussed more fully in Chapter 3.

If some nonzero vector x and its image Rx under the action of the rotation R through minimal angle $\theta(R)$ are joined by a line segment, then that line segment together with its images under all transformations in $\mathscr{H}_2^n$ bound a regular n-gon X. The subgroup $\mathscr{C}_2^n$ of rotations in $\mathscr{H}_2^n$ is the group of all rotations that leave the n-gon invariant, and $\mathscr{H}_2^n$ itself is the group of all orthogonal transformations that leave X invariant. In the case $n = 4$, $\mathscr{C}_2^4$ and $\mathscr{H}_2^4$ are the rotation group and the full orthogonal group under which the square is invariant [see Figure 2.4, where $x = (1, 1)$]. The relatively open region $F_X = F \cap X$ is a fundamental region in the square X for the group $\mathscr{H}_2^4$, in the sense discussed above.

2.3 ORTHOGONAL TRANSFORMATIONS IN THREE DIMENSIONS

We assume throughout this section that $\dim V = 3$.