Lecture Notes in Mathematics
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Ryszard Jajte

Strong Limit Theorems in Noncommutative L_2-Spaces

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- an informative introduction, perhaps with some historical remarks: it should be accessible to a reader not particularly familiar with the topic treated;
- a subject index: as a rule this is genuinely helpful for the reader.

Further remarks and relevant addresses at the back of this book.

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To my wife

PREFACE

This book is a continuation of the volume "Strong limit theorems in non-commutative probability", Lecture Notes in Mathematics 1110 (1985). It is devoted mostly to one subject: the noncommutative versions of pointwise convergence theorems in L_2-spaces in the context of von Neumann algebras.

In the classical probability and ergodic theory the almost sure convergence theorems for sequences in L_2 (over a probability space) belong to the most important and deep results of these theories. Let us mention here the individual ergodic theorems, the results on the almost sure convergence of orthogonal series, powers of contractions, martingales and iterates of conditional expectations.

The algebraic approach to quantum statistical mechanics suggests the systematic analysis of theorems just mentioned in the context of operator algebras. This is the main goal of this book. We consider a von Neumann algebra M with a faithful normal state ϕ and take $H = L_2(M,\phi)$ - the completion of M under the norm $x \to \varphi(x^*x)^{1/2}$, $x \in M$. Then we introduce a suitable notion of almost sure convergence in H (generalizing the classical one) and prove a series of theorems which can (and should) be treated as the extentions of the well-known classical results (like individual ergodic theorems, Rademacher-Menshov theorem for orthogonal series or theorem of Burkholder and Chow on the almost sure convergence of the iterates of two conditional expectations etc.).

The classical pointwise convergence theorems for sequences in L_2 are, as a rule, non-trivial extensions of much easier results concerning the convergence in L_2-norm. The same situation is in the noncommutative case. Most of the noncommutative L_2-norm versions of the analogical classical results can be rather easily obtained by a natural modification of the calssical argument. Passing to the noncommutative almost sure versions needs as a rule new methods and techniques.

Very often the algebraic approach makes much clearer the general idea which is behind the result concerning, say, real functions. At the same time the proofs provide some new tools in the theory of operator algebras. This is one of the reasons we decided to collect and prove in a systematic way the results concerning the almost sure convergence in L_2 over a von Neumann algebra.

Only very few bibliographical indications have been made in the main text of the book. More complete information concerning the subject the reader will find in the "Notes and remarks" concluding the chapters.

We hope that this book may be of some interest to probabilists and mathematical physicists concerned with applications of operator algebras to quantum statistical mechanics.

The prerequisites for reading the book are a fundamental knowledge of functional analysis and probability. Many of the results presented in the book have been discussed and also obtained during the seminar on the noncommutative probability theory in Łódź University in the years 1985-1990. I would like to thank very much all my colleagues from this seminar for many interesting and fruitful discussions.

I sincerely wish to thank Mrs Barbara Kaczmarska who took great care in the typing of the final version of the book.

Łódź, November 1990. R. Jajte

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Chapter 1

ALMOST SURE CONVERGENCE IN NONCOMMUTATIVE L_2-SPACES

1.1. Preliminaries

Throughout the book we constantly use the terminology of operator algebras. In fact, only very little knowledge of this theory is needed for reading this volume. As we mentioned in the Introduction, this book is a continuation of [50]. All necessary (and sufficient for our purpose) information concerning the operator algebras has been collected in the Appendix to [50].

Let us begin with some notation. In the sequel M will denote a σ-finite von Neumann algebra with a faithful normal state ϕ. M' denotes the commutant of M. $\mathrm{Proj}\, M$ will stand for the set of all orthogonal projections in M. M_+ will denote the cone of positive elements of M. For $p \in \mathrm{Proj}\, M$, always $p^\perp = 1 - p$. We shall write 1 for the identity operator in M. M_* denotes the predual of M.

In the whole book we shall discuss the problems concerning the Hilbert space $H = L_2(M,\phi)$ which is the completion of M under the norm $x \to \phi(x^*x)^{1/2}$ (GNS representation space for M with respect to ϕ). In the sequel we assume that M acts in a standard way, on the Hilbert space $H = L_2(M,\phi)$ with a cyclic and separating vector Ω such that $\phi(x) = (x\Omega,\Omega)$, for $x \in M$. We shall identify M with the subset $M\Omega = \{x\Omega : x \in \Omega\}$ of H. The norm in H will be denoted by $\| \ \|$, and the norm in M by $\| \ \|_\infty$.

For a $\xi \in H$ and $p \in \mathrm{Proj}\, M$ we set

$$S_{\xi,p} = \{(x_k) \subset M : \sum_{k=1}^{\infty} x_k\Omega = \xi \text{ in } H \text{ and } \sum_{k=1}^{\infty} x_k p \text{ converges in norm in } M\}$$

and

$$\|\xi\|_p = \inf \{\| \sum_{k=1}^{\infty} x_k p\|_\infty : (x_k) \in S_{\xi,p}\}$$

(with the usual convention $\inf \emptyset = +\infty$).

Obviously, for all $\xi,\eta \in H$, we have

$$\|\xi + \eta\|_p \leqq \|\xi\|_p + \|\eta\|_p$$

and for $x \in M$

$$\|x\Omega\|_p \leqq \|xp\|_\infty .$$

We adopt the following definition of the almost sure convergence in H.

1.1.1. DEFINITION. *A sequence (ξ_n) in $H = L_2(M,\phi)$ is said to be almost surely (a.s.) convergent to $\xi \in H$ if for every $\varepsilon > 0$, there exists a projection p in M such that $\Phi(1-p) < \varepsilon$ and $\|\xi_n - \xi\|_p \to 0$ as $n \to \infty$. In other words, $\xi_n \to 0$ a.s. if for every $\varepsilon > 0$, there is a $p \in \mathrm{Proj}\, M$ with $\Phi(1 - p) < \varepsilon$ and a matrix $(x_{n,k})$ with entries in M such that*

$$\sum_{k=1}^{\infty} x_{n,k}\Omega = \xi_n \quad \text{in} \quad H \quad \text{and} \quad \|\sum_{k=1}^{\infty} x_{n,k}p\|_\infty \to 0 .$$

It is easily seen that in the classical commutative case of $M = L_\infty$ (over a probability space) the convergence just defined coincides with the usual almost everywhere convergence (via Egorov´s theorem). Let us recall that for the elements of the algebra M the following kind of convergence (introduced by E.C. Lance) is mostly used. A sequence $(x_n) \subset M$ is said to be almost uniformly convergent to $x \in M$ if for every $\varepsilon > 0$ there exists a projection $p \in \mathrm{Proj}\, M$ with $\Phi(1 - p) < \varepsilon$ such that $\|(x_n - x)p\|_\infty \to 0$ as $n \to \infty$. Obviously, the almost uniform convergence implies the almost sure convergence, i.e. if x_n, $x \in M$ and $x_n \to x$ almost uniformly in M, then $x_n\Omega \to x\Omega$ almost surely in H. Let us remark that the above definition of the almost sure convergence in H can be formulated equivalently as follows: $\xi_n \to 0$ a.s. in H if for every strong neighbourhood U of the unity in M, there are a projection $p \in U$ and a matrix $(x_{n,k})$ with entries in M such that $\sum_{k=1}^{\infty} x_{n,k}\Omega = \xi_n$ $(n = 1,2,\ldots)$ in H and $\|\Sigma x_{n,k}p\|_\infty \to 0$ as $n \to \infty$. This means that the a.s. convergence in H depends only on M and cyclic element $\Omega \in H$.

To end this section let us compare our notion of the almost sure convergence in H introduced in Definition 1.1.1 with another given by M.S. Goldstein [36]. He uses the following notion of the a.s. convergence in H. Namely, for ξ_n, $\xi \in H$, $\xi_n \to \xi$ a.s. in the sense of Goldstein if, for every $\varepsilon > 0$, there exists a projection $p \in \mathrm{Proj}\, M$

and $(x_k) \subset M$ such that $\Phi(1-p) < \varepsilon$, $p(\xi_n - \xi) = x_n\Omega$, for n large enough, and $\|x_n\|_\infty \to 0$ as $n \to \infty$.

The notion of the a.s. convergence in H introduced in Definition 1.1.1 seems to be more natural (though) in the classical case of $M = L_\infty$ (over a probability space) both notions coincide). The reason for that we prefer our definition is, roughly speaking, the following. The almost sure convergence has a very clear and nice interpretation on the ground of the classical probability and statistics. It says that we have the convergence of practically all (with probability one) realizations of a suitable stochastic process. In quantum mechanics the interpretation is entirely different (we have no trajectories of a process in the sense of the classical theory). In quantum probability we are rather interested to be close to the uniform convergence of operators (observables). Let us mention here that in connection with the characterization of C*-algebras by Gelfand and Naymark, Segal argued that the uniform convergence of observables has a direct physical interpretation, while weak convergence has rather analytical meaning. This opinion is not common and rather disputable but at least we can say that the uniform convergence is the best one not only from analytical but also from the physical point of view. "Close to the uniform convergence" means in our context uniform convergence on large subspaces, where large subspacess are just those for which the values of a state on the corresponding orthogonal projections are close to one. Our definition of the a.s. convergence seems to fit better to this interpretation then the notion proposed by Goldstein because we are interested in what happens on the (large) subspaces. That is why the projection p appearing in our definition of the a.s. convergence is put on the right side of the operators (observables) not on the left side of them. Clearly, both definitions coincide for selfadjoint observables i.e. when we consider only the selfadjoint part M^{sa} of M and its completion under the norm $\| \ \|$.

1.2. Auxiliary results

In section we collect a few results concerning some simple properties of the almost sure convergence in H. In the sequel, for $x \in M$, we put $|x|^2 = x^*x$. Let us note the following inequality

1.2.1. LEMMA. *Let* $\alpha_1, \alpha_2, \ldots, \alpha_N$ *be complex numbers, and* $x_1, \ldots, x_N \in M$. *Then*

$$|\sum_{i=1}^{N} \alpha_i x_i|^2 \leqq \sum_{i=1}^{N} |\alpha_i|^2 \sum_{i=1}^{N} |x_i|^2.$$

Proof. This easily follow by induction from the inequality $x^*x + y^*y \leqq x^*x^* + y^*y^*$ $(x, y \in M)$. ∎

We call $\alpha \in L(M)$ a Schwarz map if α satisfies the inequality $|\alpha(x)|^2 \leqq \alpha(|x|^2)$, for $x \in M$. Note that α is then necessarily a contraction in M.

A map $\alpha \in L(M)$ is said to be ϕ-contractive if $\phi(\alpha x) \leqq \phi(x)$, for all $x \in M_+$. A normal ϕ-contractive Schwarz map in M will be called a kernel.

Let β_o be a kernel in M. Then one can extend β_o (in a unique way) to a contraction β in H. Namely, we put $\beta(x\Omega) = \beta(x)\Omega$, for $x \in M$, and then extend the obtained contraction from $M\Omega$ to the whole H by continuity. In this case we shall say that the contraction β in H is generated by the kernel β_o in M.

The most important examples of kernels are ϕ-preserving *-endomorphisms of M (in particular isomorphisms) and ϕ-preserving conditional expectations. They generate isometries (in particular unitary operators) and orthogonal projections in H, respectively.

For a kernel $\alpha : M \to M$, we denote by $\alpha' : M' \to M'$ the dual of α. In particular, we have

$$(\alpha(x)y\Omega, \Omega) = (x\alpha'(y)\Omega, \Omega)$$

and

$$(\alpha'(y)\Omega, \Omega) \leqq (y\Omega, \Omega) \qquad \text{for } x \in M, \quad y \in M'$$

(for more details see f. ex. [50], p. 14).

Now, we shall prove a result concerning the continuity of some kernels with respect to the a.s. convergence.

1.2.2. PROPOSITION. *Let* β_o *be a* ϕ*-preserving* **-endomorphism of* M. *Denote by* β *the contraction in* H *generated by* β_o. *Then, for every* $(\xi_n) \subset H$, $\xi_n \to 0$ *a.s. in* H *implies* $\beta\xi_n \to 0$ *a.s. in* H.

Proof. Assume that $\xi_n \to 0$ a.s. in H. Let $\varepsilon > 0$ be given. Then there exists a projection $p \in \text{Proj } M$ with $\Phi(1-p) < \varepsilon/2$ and $(x_{n,k}) \subset M$ such that $\Sigma x_{n,k}\Omega = \xi_n$ in H, the series $\sum_k x_{n,k}p$ converges in norm in M and $\|\sum_k x_{n,k}p\|_\infty \to 0$ as $n \to \infty$. By the properties of β, we have $\beta\xi_n = \Sigma\beta_o(x_{n,k})\Omega$ in H and $\|\Sigma\beta_o(x_{n,k})\beta_o(p)\|_\infty \to 0$ as $n \to \infty$. Let $\beta_o(p) = \int_o^1 \lambda e(d\lambda)$ be the spectral representation of the operator $\beta_o(p)$. Put $q = e([1/2,1])$. Then we have $q = \beta_o(p)a$ $a = \int_{1/2}^1 1/\lambda\, e(d\lambda) \in M$. Then also $\|\sum_k \beta_o(x_{n,k})q\|_\infty \to 0$. Moreover, $\beta_o(p) \leq q + \frac{1}{2}(1-q) = \frac{1}{2}(1+q)$ and, consequently, $\phi(q) \geq 1 - \varepsilon$. Summing up, for $\varepsilon > 0$ there is a $q \in \text{Proj } M$ with $\phi(1-q) < \varepsilon$ and a matrix $(y_{n,k}) = (\beta_o(x_{n,k})) \subset M$ such that $\beta\xi_n = \sum_k y_{n,k}\Omega$ in H and $\|\sum_k y_{n,k}q\|_\infty \to 0$ which means that $\beta\xi_n \to 0$ a.s. and completes the proof. ▪

1.2.3. LEMMA. *For* $(\xi_n) \subset H$, $\sum_n \|\xi_n\|^2 < \infty$ *implies* $\xi_n \to 0$ *a.s. in* H.

Proof. Let $\varepsilon > 0$. We shall find $p \in \text{Proj } M$ such that $\phi(1-p) < \varepsilon$ and $\|\xi_n\|_p \to 0$ as $n \to \infty$. Let $(x_{n,k})$ be a matrix with entries in M such that $\xi_n = \sum_{k=1}^{\infty} x_{n,k}\Omega$ and $\|x_{n,k}\Omega\| \leq 2^{-k+1}\|\xi_n\|$, for $n,k = 1, 2,\ldots$. Take (δ_n) with $\sum_n \delta_n^{-1}\|\xi_n\|^2 < \varepsilon/4$. Then $\sum_{k=1}^{\infty}\sum_{k=1}^{\infty}$

$$\sum_{n=1}^{\infty}\ \sum_{k=1}^{\infty} \delta_n 2^{-k}\phi(|x_{n,k}|^2) \leq 2 \sum_{n=1}^{\infty} \delta_n^{-1}\|\xi_n\|^2 < \varepsilon/2.$$

By Goldstein's maximal ergodic theorem ([36], [50], see also section 2.2 of Chapter 2) there exists a projection $p \in \text{Proj } M$ such that

$$\phi(1-p) \leq 2 \sum_{k=1}^{\infty} \delta_k^{-1}\|\xi_k\|^2 < \varepsilon$$

and $\|p|x_{n,k}|^2p\|_\infty < 2^{-k+2}\delta_n$ $(n,k = 1,2,\ldots)$.

Moreover, since $\xi_n = \sum_k x_{n,k}\Omega$, we have

$$\|\xi_n\|_p \leq \sum_{k=1}^{\infty} \|x_{n,k}p\|_\infty \leq \sum_{k=1}^{\infty} \|p|x_{n,k}|^2p\|_\infty^{1/2} \leq 5\delta_n^{1/2} \to 0,$$

which means that $\xi_n \to 0$ a.s. in H. ▪

1.2.4 (KRONECKER'S LEMMA). *Let* $(\xi_n) \subset H$ *and* $\sum_{k=1}^{n} \xi_k \to \eta$ *a.s.* *Then* $n^{-1} \sum_{k=1}^{n} k\xi_k \to 0$ *a.s.*

Proof. Let us remark first that, for every sequence $(\eta_n) \subset H$ and every $p \in \mathrm{Proj}\, M$, $\|\eta_n\|_p \to 0$ implies $\|n^{-1} \sum_{k=1}^{n} \eta_k\|_p \to 0$. Indeed, let $\|\eta_n\|_p \to 0$. Then there exists a matrix $(x_{n,k})$ with entries in M such that $\sum_{k=1}^{\infty} x_{n,k}\Omega = \eta_n$ $(n = 1,2,\ldots)$ and $\|\sum_{k=1}^{\infty} x_{n,k}p\| \to 0$ as $n \to \infty$. Thus

$$\frac{1}{n} \sum_{k=1}^{n} \eta_k = \frac{1}{n} \sum_{k=1}^{n} \sum_{l=1}^{\infty} x_{k,l}\Omega$$

and

$$\|\frac{1}{n} \sum_{k=1}^{n} \sum_{l=1}^{\infty} x_{k,l}\, p\|_\infty \le \frac{1}{n} \sum_{k=1}^{n} \|\sum_{l=1}^{\infty} x_{k,l}p\|_\infty \to 0$$

which means that $\|\frac{1}{n} \sum_{k=1}^{n} \eta_k\|_p \to 0$.

Let us put $\eta_n = \sum_{k=1}^{n} \xi_k$. By the assumptions, for every $\varepsilon > 0$, there exists $p \in \mathrm{Proj}\, M$ such that $\Phi(1 - p) < \varepsilon$ and $\|\eta_n - \eta\|_p \to 0$. Then $\|\frac{1}{n} \sum_{k=1}^{n} (\eta_k - \eta)\|_p \to 0$. Consequently,

$$\|\frac{1}{n} \sum_{k=1}^{n} k\xi_k\|_p = \|\eta_n - \frac{1}{n} \sum_{k=1}^{n} \eta_{k-1}\|_p$$

$$\le \|\eta_n - \eta\|_p + \|\frac{1}{n} \sum_{k=1}^{n} (\eta_{k-1} - \eta)\|_p \to 0,$$

which ends the proof. ▪

The following result is a noncommutative version of the well-known theorem of Revesz [94].

1.2.5. THEOREM. Let $(\xi_n) \subset H$ *such that* $\sup_n \|\xi_n\| < \infty$. *Then there exists an increasing sequence of positive integers* (n_k) *and* $\eta \in H$ *such that* $k^{-1}(\xi_{n_1} + \ldots + \xi_{n_k}) \to \eta$ *a.s. in* H.

Proof. Let $\|\xi_i\| \leq C$ $(i = 1,2,\ldots)$. By the weak compactness of (ξ_i), there is a sequence (n_s) of positive integers and $\eta \in H$ such that

$$\langle \xi_{n_s}, \psi\rangle \to \langle \eta, \psi\rangle \quad \text{as} \quad s \to \infty,$$

for all $\psi \in H$.

Put $\xi_{n_s} - \eta = \zeta_s$. Then $\zeta_k \to 0$ weakly. Therefore there is a subsequence (ζ_{n_k}) of (ζ_k) such that

$$|\langle \zeta_{n_k}, \zeta_{n_s}\rangle| \leq \frac{1}{2^k}, \qquad \begin{array}{l}\text{for } s = 1,2,\ldots,k-1;\\ \text{and } k = 1,2,\ldots .\end{array}$$

Put

$$\sigma_k = k^{-1}(\zeta_{n_1} + \zeta_{n_2} + \zeta_{n_k}).$$

It is easily seen that $\|\sigma_k\|^2 = O(k^{-1}))$ and, consequently,

$$\sum_{k=1}^{\infty} \|\sigma_{k^2}\|^2 < \infty.$$

For $k^2 \leq N < (k+1)^2$, we have

$$\sigma_N = \alpha_{N,k} + \beta_{N,k},$$

where

$$\alpha_{N,k} = N^{-1}k^2\sigma_{k^2}$$

and

$$\beta_{N,k} = N^{-1}(\zeta_{n_{(k^2+1)}} + \zeta_{n_{(k^2+2)}} + \ldots + \zeta_{n_N}).$$

It is not difficult to show that

$$\sum_{k=1}^{\infty} \sum_{k^2 \leq N < (k+1)^2} \|\beta_{N,k}\|^2 < \infty.$$

The last inequality and $\sum_{k=1}^{\infty} \|\sigma_{k^2}\|^2 < \infty$ imply that, for every $\varepsilon > 0$,