Lecture Notes in Mathematics

Edited by A. Dold and B. Eckmann

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Michael Taylor

Pseudo Differential Operators

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LECTURES ON PSEUDO DIFFERENTIAL OPERATORS

INTRODUCTION: These notes are based on the lectures I gave in partial differential equations at the University of Michigan during the winter semester of 1972, with some extensions. References to further work have been added at the end of Chapters III, IV, and V, and a few exercises have been thrown in, in addition to those thrown out in class.

The students to whom these lectures were addressed were assumed to have knowledge of elementary functional analysis, the Fourier transform, distribution theory, and Sobolev spaces, and such tools are used without comment. We refer the reader especially to Yosida [85] for the background material. The last section of Chapter I also relies on the basic results of C* algebra theory, and the reader who doesn't like functional analysis might have to skip this section on first reading. Beyond that, we have tried to make these notes self-contained.

This is not to say that these notes constitute a self contained introduction to the subject of partial differential equations, and the beginning student would have to see the material in several of the books we have mentioned in the references, especially, [13], [25], [31], and [1], [52], [54], [55], [59], in order to get a good idea of what the subject is about. What we do here is develop one tool, the calculus of pseudo differential operators, and apply it to several of the main problems of partial differential equations.

We begin in Chapter I by describing the earliest sort of singular integrals on the circle investigated by Poincare, Hilbert, and others, and an application to the oblique derivative problem on the disc. In the second Chapter we introduce the modern calculus of pseudo differential operators, developed by Kohn and Nirenberg, Lax, Hörmander, Kumano-Go, and others, and in Chapter III we apply these results to obtain interior regularity results for elliptic and hypoelliptic operators. The next two chapters are devoted to the main topics of classical PDE, the initial value problem for hyperbolic and parabolic equations, and boundary value problems for elliptic equations. We give a unified treatment of these topics, and Gårding's inequality plays a crucial role here in passing from formal properties of symbols to the energy inequalities and other *a priori* inequalities needed for various results on existence and regularity.

In Chapter VI we cover some recent work of Hörmander on wave front sets and the propagation of singularities of solutions to partial differential equations. Applications are given to local existence of solutions to PDE's and to an exponential decay result. The proof of the main result on propagation of singularities requires the sharp Gårding inequality which, following Kumano-Go, we prove in the last chapter of these notes.

One important topic we have not included is Uniqueness in the Cauchy problem. We recommend that the reader consult [91].

It is a pleasure to thank Eric Bedford, whose classroom notes greatly aided the preparation of these notes, and Professor Jeff Rauch for some interesting conversations, especially relating to hyperbolic equations.

CHAPTER I. SINGULAR INTEGRAL OPERATORS ON THE CIRCLE

The basic singular integral operator with which we will be concerned here can be described as follows.

If $u \in L^2(S^1)$, write $u(\phi) = \sum_{n=-\infty}^{\infty} a_n e^{in\phi}$, and define

$$Pu = \sum_{n=0}^{\infty} a_n e^{in\phi}$$

In order to interpret the operator P, which is clearly a continuous orthogonal projection on $L^2(S^1)$, as a singular integral operator, consider the Cauchy integral

$$Tu(z) = \frac{1}{2\pi i} \int_{S^1} \frac{u(\zeta)}{\zeta - z} d\zeta \qquad (|z| < 1)$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{u(\phi)}{e^{i\phi} - z} e^{i\phi} d\phi$$

We can rewrite this as

$$Tu(re^{i\theta}) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{u(\phi)}{1 - re^{i(\theta-\phi)}} d\phi$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{u(\phi) - u(\theta)}{1 - re^{i(\theta-\phi)}} d\phi + u(\theta)$$

If $u \in C^1(S^1)$, we can pass to the limit as $r \to 1$ and obtain

$$\lim T u(re^{i\theta}) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{u(\phi) - u(\theta)}{1 - e^{i(\theta-\phi)}} d\phi + u(\theta)$$

$$= \lim_{\varepsilon\to 0} \frac{1}{2\pi} \int_{S^1\setminus I_\varepsilon(\theta)} \frac{u(\phi) - u(\theta)}{1-e^{i(\theta-\phi)}} d\phi + u(\theta)$$

$$= \lim_{\varepsilon\to 0} \frac{1}{2\pi i} \int_{S^1\setminus I_\varepsilon(\theta)} \frac{u(\zeta) - u(\theta)}{\zeta-e^{\theta}} d\zeta + u(\theta)$$

$$= \frac{1}{2\pi i} \text{PV} \int_{S^1} \frac{u(\zeta)}{\zeta-e^{i\theta}} d\zeta + \frac{1}{2} u(\theta)$$

where $I_\varepsilon(\theta) = (\theta - \varepsilon, \theta + \varepsilon)$.

Since it is easy to verify that $\lim_{r\uparrow 1} Tu(re^{i\theta}) = Pu$ for $u \in C^1(S^1)$,*

it follows that $Pu = \frac{1}{2} Hu + \frac{1}{2} u$ where

$$Hu(e^{i\theta}) = \frac{1}{\pi i} \text{PV} \int_{S^1} \frac{u(\zeta)}{\zeta-e^{i\theta}} d\zeta$$

The singular integral operator H is called the Hilbert transform. The formula we have just derived shows that H extends to a continuous linear operator on $L^2(S^1)$.

Exercise 1. Find the Fourier series representation of H. Prove that H is a unitary operator on $L^2(S^1)$ and $H^2 = I$.

* (using the residue theorem, the reader should check that, if $u_k = e^{ik\theta}$, then $\lim Tu_k(re^{i\theta}) = u_k$ if $k \geq 0$, 0 if $k < 0$.)

§1. The algebra of singular integral operators.

Definition: The algebra $\mathfrak{A}$ of singular integral operators on S^1 is the norm closed algebra of operators on $L^2(S^1)$ generated by:

(1) P

(2) multiplication by $a \in C(S^1)$

(3) $\mathfrak{C}$, the set of compact operators

Actually (3) is redundant, but we shall not prove this fact, nor make use of it.

Theorem 1: If $A, B \in \mathfrak{A}$, then $[A,B] = AB - BA \in \mathfrak{C}$.

Proof: It suffices to show that $aP - PA \in \mathfrak{C}$ if $a \in C(S^1)$. Suppose that $a = e^{im\phi}$, $f = \sum_{n=-\infty}^{\infty} a_n e^{in\phi}$.

Then $aPf = e^{im\phi} \sum_{n=0}^{\infty} a_n e^{in\phi} = \sum_{n=m}^{\infty} a_{n-m} e^{in\phi}$, and

$$Paf = P\left(\sum_{n=-\infty}^{\infty} a_{n-m} e^{in\phi}\right) = \sum_{n=0}^{\infty} a_{n-m} e^{in\phi} .$$

The $[a,P]f = \sum_{n=0}^{m-1} a_{n-m} e^{in\phi}$. Hence $[a,P]$ in an operator with finite dimensional range, and therefore is compact. Since trigonometric polynomials are dense in $C(S^1)$, the result holds for all $a \in C(S^1)$, because $\mathfrak{C}$ is norm closed.

This theorem, which says that $\mathfrak{A}$ is commutative, modulo $\mathfrak{C}$ is important in that it enables us to give a nice conditon that an operator in $\mathfrak{A}$ be Fredholm. For the moment, consider an operator $T \in \mathfrak{A}$ of the form

$$T = aP + b(1-P) + K \quad K \in \mathcal{C}$$

In section 3 we shall show that every $T \in \mathfrak{A}$ is of this form, but we won't need this, since all singular integral operators one encounters are automatically constructed in this form. For such a T, we tentatively define the symbol σ_T of T, as a function on $S^1 \times \mathbb{Z}_2$ by

$$\sigma_T(\phi,1) = a(\phi)$$

$$\sigma_T(\phi,-1) = b(\phi)$$

We show that σ_T is indeed well defined.

Lemma 1: If $aP + b(1-P) \in \mathcal{C}$, then $a \equiv b \equiv 0$.

Proof: $(aP + b(1-P))P = aP \in \mathcal{C}$, since $P^2 = P$.
Then $|a|^2 P \in \mathcal{C}$.
If $U_\phi f(\theta) = f(\phi-\theta)$, the map $\phi \to U_\phi |a|^2 PU_{-\phi}$ is continuous in the uniform operator topology, since

$$U_\phi |a|^2 PU_{-\phi} f(\theta) = |a(\theta-\phi)|^2 Pf(\theta)$$. Thus

$$\frac{1}{2\pi} \int_0^{2\pi} U_\phi |a|^2 PU_{-\phi}\, d\phi = \|a\|_2^2 P \in \mathcal{C},$$ which forces $a \equiv 0$.

Similary we obtain $b \equiv 0$.

Theorem 2: If $T = aP + b(1-P) + K_1$, and $W = \alpha P + \beta(1-P) + K_2$ then $\sigma_T \sigma_W = \sigma_{TW}$.

Proof: This is immediate from the computation

$$TW = (aP + b(1-P) + K_1)\ (\alpha P + \beta(1-P) + K_2)$$
$$= a\alpha P^2 + b\beta(1-P)^2 + K_3$$
$$= a\alpha P + b\beta(1-P) + K_3 \quad .$$

Recall that a linear operator $T \in \mathcal{L}(L^2)$ is called Fredholm if

(1) $R(T)$ is closed

(2) $\dim \ker T < \infty$

(3) $\dim \operatorname{coker} T < \infty$.

The reader should also recall the following important result from the Riesz theory of compact operators (see [64], Chap. VII)

Proposition: $T \in \mathcal{L}(L^2)$ is Fredholm if and only if there exists $U \in \mathcal{L}(L^2)$, called a Fredholm inverse of T, such that $TU = I + K_1$, and $UT = I + K_2$, where K_1 , K_2 are compact.

The following Fredholm property of singular integral operators is now immediate.

Theorem 3: Let $T = aP + b(1-P) + K$. Then T is Fredholm if σ_T is nowhere vanishing.

Proof. Let $U = \frac{1}{a} P + \frac{1}{b} (1-P)$. Then $\sigma_{TU} = \sigma_{UT} \equiv 1$, so U is a Fredholm inverse of T .

In section 3 we shall show that this conditon on σ_T is also necessary for T to be Fredholm on $L^2(S')$.

The problem of how to define the symbol of a singular integral operator on a multidimensional space took quite some timé in being solved. Mikhlin defined a symbol in 1936. This symbol was elucidated by Calderon and Zygmund in their important works in the early 1950's. It was Lax who suggested a Fourier series representation to treat multidimensional singular integrals and the Fourier integral representation used by Kohn and Nirenberg is the one we shall use in the next chapter.

Exercise: Let $Tf(x) = \frac{1}{\pi i} \, PV \int_{S^1} \frac{a(x,y)}{y-x} f(y)\, dy$, where $a \in C^\infty(S^1 \times S^1)$. Show that

$$Tf = bHf + Kf$$

where $b(x) = a(x,x)$ and $K: H^s \to H^s$ is compact, for all s.

§2. The oblique derivative problem.

Here we discuss one application of the algebra of singular integrals developed above. For further applications we refer the reader to Mikhlin [58] and Muskhelishvili [60] The problem we consider is the oblique derivative problem for functions harmonic on the disc: given $g \in C(S^1)$, find u harmonic on $B = \{z \in \mathbb{C} : |z| < 1\}$ such that

(1) $$\beta u = a \frac{\partial}{\partial r} u + b \frac{\partial}{\partial \phi} u + c u \Big|_{S^1} = g .$$

The way we handle this is as follows. First, the Dirichlet problem

$$\Delta u = 0 \quad \text{in} \quad B$$

$$u\Big|_{S^1} = f$$

can be solved explicitly by $u = PIf$, the Poisson integral of f. To derive this Poisson integral representation, write

$$f = \sum_{n=-\infty}^{\infty} a_n e^{in\phi} . \quad \text{Then}$$

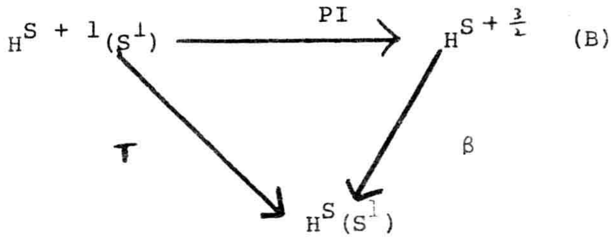
$$PIf(re^{i\phi}) = \sum_{n-\infty}^{\infty} a_n r^{|n|} e^{in\phi}$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{1 - r^2}{1+r^2 - 2r \cos(\phi-\theta)} f(\theta)\, d\theta$$

The reader can verify as an exercise that PIf does the trick.

Exercise 2. Prove that if $s \geq -\frac{1}{2}$ and $f \in H^s(S)$, then $PIf \in H^{S+\frac{1}{2}}(B)$. (On first reading, don't take this exercise too seriously.)

In view of the fact that restriction to S^1 maps $H^{\tau}(B)$ onto $H^{\tau-\frac{1}{2}}(S^1)$ for $\tau > \frac{1}{2}$, we have the following commutative diagram with $S > 0$.



where we define $T = \beta \circ PI$. Hence we can solve (1) by setting $u = PI\, h$ if $Th = g$. Hence solving (1) is equivalent to inverting T.

More generally, we are interested in when problem (1) is Fredholm, in the sense that it can be solved provided g satisfies a certain finite number of linear conditions, and the set of u satisfying $\Delta u = 0$, $\beta u = 0$ should be finite dimensional.

This is equivalent to asking when is T Fredholm, which is right up our alley, since we will now write T as a pseudo differential operator. We have

$$u = PIf = \sum_{n-\infty}^{\infty} a_n r^{|n|} e^{in\phi}$$

$$\frac{\partial}{\partial \phi} u \Big|_{S^1} = \frac{\partial}{\partial \phi} f = \sum_{n-\infty}^{\infty} in\, a_n e^{in\phi}$$

and $\frac{\partial}{\partial r} u \Big|_{S'} = \sum_{n=-\infty}^{\infty} |n|\, a_n e^{in\phi}$

Now let us define an operator Λ by

$$\Lambda\left(\sum_{n=-\infty}^{\infty} a_n e^{in\phi}\right) = \sum_{n=-\infty}^{\infty} (1+|n|)\, a_n e^{in\phi}$$

Exercise 3. Prove that $\Lambda : H^S(S^1) \to H^{S-1}(S^1)$ isomorphically, for each real S.

It follows that $T = a\frac{\partial}{\partial r} + b\frac{\partial}{\partial\phi} + C = a(\Lambda - 1) + b\frac{\partial}{\partial\phi} + C$

$$= (a + b\frac{\partial}{\partial\phi}\Lambda^{-1} + (c-a)\Lambda^{-1})\Lambda \quad .$$

Since Λ is an isomorphism of $H^{S+1}(S^1)$ onto $H^S(S^1)$ it follows that $T:H^{S+1}(S^1) \to H^S(S^1)$ is Fredholm if and only if $S = (a + b\frac{\partial}{\partial\phi}\Lambda^{-1} + (c-a)\Lambda^{-1}) : H^S \to H^S$ is Fredholm.

Exercise 4. $\Lambda^{-1} : H^S \to H^S$ is compact, for each real S.

Now we compute that $\frac{\partial}{\partial\phi}\Lambda^{-1}(\sum a_n e^{in\phi})$

$$= i\sum \frac{n}{1+|n|} a_n e^{in\phi} = i(2P-1)(\sum a_n e^{in\phi}) - \sum \frac{a_n}{1+|n|} e^{in\phi}$$

or $\frac{\partial}{\partial\phi}\Lambda^{-1} f = i(2P-1)f + Kf$, where K is compact on each H^S. Thus we see that S is a singular integral, given by

$$S = a + ib(2P-1) + K$$
$$= (a+ib)P + (a-ib)(1-P) + K$$