Lecture Notes in Mathematics

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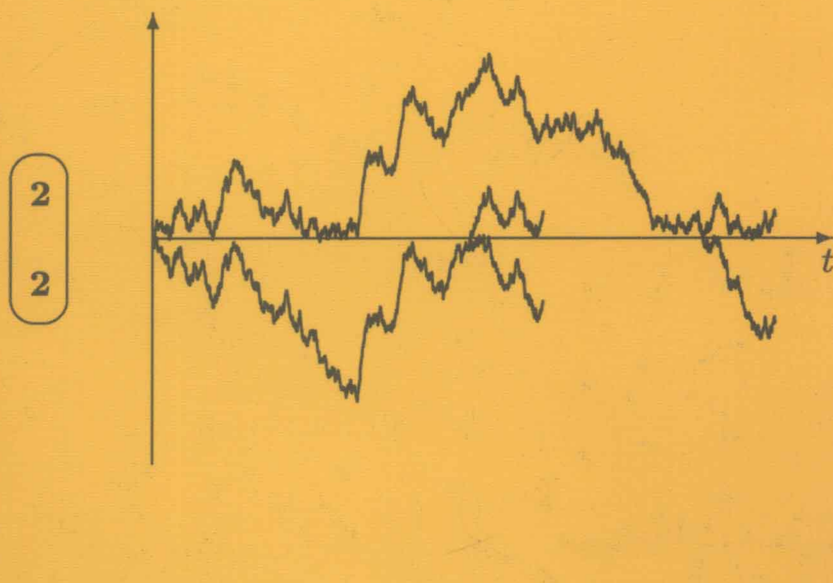
Alexander S. Cherny
Hans-Jürgen Engelbert

Singular Stochastic Differential Equations



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Preface

We consider one-dimensional homogeneous stochastic differential equations of the form

$$dX_t = b(X_t)dt + \sigma(X_t)dB_t, \quad X_0 = x_0, \tag{$*$}$$

where b and σ are supposed to be measurable functions and $\sigma \neq 0$.

There is a rich theory studying the existence and the uniqueness of solutions of these (and more general) stochastic differential equations. For equations of the form $(*)$, one of the best sufficient conditions is that the function $(1+|b|)/\sigma^2$ should be locally integrable on the real line. However, both in theory and in practice one often comes across equations that do not satisfy this condition. The use of such equations is necessary, in particular, if we want a solution to be positive. In this monograph, these equations are called *singular stochastic differential equations.* A typical example of such an equation is the stochastic differential equation for a geometric Brownian motion.

A point $d \in \mathbb{R}$, at which the function $(1+|b|)/\sigma^2$ is not locally integrable, is called in this monograph a *singular* point. We explain why these points are indeed "singular". For the *isolated singular points*, we perform a complete qualitative classification. According to this classification, an isolated singular point can have one of 48 possible types. The type of a point is easily computed through the coefficients b and σ. The classification allows one to find out whether a solution can leave an isolated singular point, whether it can reach this point, whether it can be extended after having reached this point, and so on.

It turns out that the isolated singular points of 44 types do not disturb the uniqueness of a solution and only the isolated singular points of the remaining 4 types disturb uniqueness. These points are called here the *branch points.* There exists a large amount of "bad" solutions (for instance, non-Markov solutions) in the neighbourhood of a branch point. Discovering the branch points is one of the most interesting consequences of the constructed classification.

The monograph also includes an overview of the basic definitions and facts related to the stochastic differential equations (different types of existence and uniqueness, martingale problems, solutions up to a random time, etc.) as well as a number of important examples.

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Introduction

The basis of the theory of diffusion processes was formed by Kolmogorov [30] (the Chapman–Kolmogorov equation, forward and backward partial differential equations). This theory was further developed in a series of papers by Feller (see, for example, [16], [17]).

Both Kolmogorov and Feller considered diffusion processes from the point of view of their finite-dimensional distributions. Itô [24], [25] proposed an approach to the "pathwise" construction of diffusion processes. He introduced the notion of a stochastic differential equation (abbreviated below as *SDE*). At about the same time and independently of Itô, SDEs were considered by Gikhman [18], [19]. Stroock and Varadhan [44], [45] introduced the notion of a martingale problem that is closely connected with the notion of a SDE.

Many investigations were devoted to the problems of existence, uniqueness, and properties of solutions of SDEs. Sufficient conditions for existence and uniqueness were obtained by Girsanov [21], Itô [25], Krylov [31], [32], Skorokhod [42], Stroock and Varadhan [44], Zvonkin [49], and others. The evolution of the theory has shown that it is reasonable to introduce different types of solutions (weak and strong solutions) and different types of uniqueness (uniqueness in law and pathwise uniqueness); see Liptser and Shiryaev [33], Yamada and Watanabe [48], Zvonkin and Krylov [50]. More information on SDEs and their applications can be found in the books [20], [23], [28, Ch. 18], [29, Ch. 5], [33, Ch. IV], [36], [38, Ch. IX], [39, Ch. V], [45].

For one-dimensional homogeneous SDEs, i.e., the SDEs of the form

$$dX_t = b(X_t)dt + \sigma(X_t)dB_t, \quad X_0 = x_0, \tag{1}$$

one of the weakest sufficient conditions for weak existence and uniqueness in law was obtained by Engelbert and Schmidt [12]–[15]. (In the case, where $b = 0$, there exist even necessary and sufficient conditions; see the paper [12] by Engelbert and Schmidt and the paper [1] by Assing and Senf.) Engelbert and Schmidt proved that if $\sigma(x) \neq 0$ for any $x \in \mathbb{R}$ and

$$\frac{1 + |b|}{\sigma^2} \in L^1_{\text{loc}}(\mathbb{R}), \tag{2}$$

then there exists a unique solution of (1). (More precisely, there exists a unique solution defined up to the time of explosion.)

Condition (2) is rather weak. Nevertheless, SDEs that do not satisfy this condition often arise in theory and in practice. Such are, for instance, the SDE for a geometric Brownian motion

$$dX_t = \mu X_t dt + \sigma X_t dB_t, \quad X_0 = x_0$$

(the Black-Scholes model!) and the SDE for a δ-dimensional Bessel process ($\delta > 1$):

$$dX_t = \frac{\delta - 1}{2X_t} dt + dB_t, \quad X_0 = x_0.$$

In practice, SDEs that do not satisfy (2) arise, for example, in the following situation. Suppose that we model some process as a solution of (1). Assume that this process is positive by its nature (for instance, this is the price of a stock or the size of a population). Then a SDE used to model such a process should *not* satisfy condition (2). The reason is as follows. If condition (2) is satisfied, then, for any $a \in \mathbb{R}$, the solution reaches the level a with strictly positive probability. (This follows from the results of Engelbert and Schmidt.)

The SDEs that do not satisfy condition (2) are called in this monograph *singular SDEs*. The study of these equations is the subject of the monograph. We investigate three main problems:

(i) *Does there exist a solution of* (1)*?*

(ii) *Is it unique?*

(iii) *What is the qualitative behaviour of a solution?*

In order to investigate singular SDEs, we introduce the following definition. A point $d \in \mathbb{R}$ is called a *singular point* for SDE (1) if

$$\frac{1 + |b|}{\sigma^2} \notin L^1_{\text{loc}}(d).$$

We always assume that $\sigma(x) \neq 0$ for any $x \in \mathbb{R}$. This is motivated by the desire to exclude solutions which have sojourn time in any single point. (Indeed, it is easy to verify that if $\sigma \neq 0$ at a point $z \in \mathbb{R}$, then any solution of (1) spends no time at z. This, in turn, implies that any solution of (1) also solves the SDE with the same drift and the diffusion coefficient $\sigma - \sigma(z) I_{\{z\}}$. "Conversely", if $\sigma = 0$ at a point $z \in \mathbb{R}$ and a solution of (1) spends no time at z, then, for any $\eta \in \mathbb{R}$, it also solves the SDE with the same drift and the diffusion coefficient $\sigma + \eta I_{\{z\}}$.)

The first question that arises in connection with this definition is: Why are these points indeed "singular"? The answer is given in Section 2.1, where we explain the qualitative difference between the singular points and the regular points in terms of the behaviour of solutions.

Using the above terminology, we can say that a SDE is singular if and only if the set of its singular points is nonempty. It is worth noting that in practice one often comes across SDEs that have only one singular point (usually, it is zero). Thus, the most important subclass of singular points is formed by the *isolated singular points*. (We call $d \in \mathbb{R}$ an isolated singular point if d is singular and there exists a deleted neighbourhood of d that consists of regular points.)

In this monograph, we perform a complete qualitative classification of the isolated singular points. The classification shows whether a solution can leave an isolated singular point, whether it can reach this point, whether it can be extended after having reached this point, and so on. According to this classification, an isolated singular point can have one of 48 possible types. The type of a point is easily computed through the coefficients b and σ. The constructed classification may be viewed as a counterpart (for SDEs) of Feller's classification of boundary behaviour of continuous strong Markov processes.

The monograph is arranged as follows.

Chapter 1 is an overview of basic definitions and facts related to SDEs, more precisely, to the problems of the existence and the uniqueness of solutions. In particular, we describe the relationship between different types of existence and uniqueness (see Figure 1.1 on p. 8) and cite some classical conditions that guarantee existence and uniqueness. This chapter also includes several important examples of SDEs. Moreover, we characterize all the possible combinations of existence and uniqueness (see Table 1.1 on p. 18).

In Chapter 2, we introduce the notion of a singular point and give the arguments why these points are indeed "singular". Then we study the existence, the uniqueness, and the qualitative behaviour of a solution in the right-hand neighbourhood of an isolated singular point. This leads to the one-sided classification of isolated singular points. According to this classification, an isolated singular point can have one of 7 possible *right types* (see Figure 2.2 on p. 39).

In Chapter 3, we investigate the existence, the uniqueness, and the qualitative behaviour of a solution in the two-sided neighbourhood of an isolated singular point. We consider the effects brought by the combination of right and left types. Since there exist 7 possible right types and 7 possible left types, there are 49 feasible combinations. One of these combinations corresponds to a regular point, and therefore, an isolated singular point can have one of 48 possible types. It turns out that the isolated singular points of only 4 types can disturb the uniqueness of a solution. We call them the *branch points* and characterize all the strong Markov solutions in the neighbourhood of such a point.

In Chapter 4, we investigate the behaviour of a solution "in the neighbourhood of $+\infty$". This leads to the classification at infinity. According to this classification, $+\infty$ can have one of 3 possible types (see Figure 4.1 on p. 83). The classification shows, in particular, whether a solution can explode into $+\infty$. Thus, the well known Feller's test for explosions is a consequence of this classification.

All the results of Chapters 2 and 3 apply to local solutions, i.e., *solutions up to a random time* (this concept is introduced in Chapter 1). In the second part of Chapter 4, we use the obtained results to study the existence, the uniqueness, and the qualitative behaviour of global solutions, i.e., solutions in the classical sense. This is done for the SDEs that have no more than one singular point (see Tables 4.1–4.3 on pp. 88, 89).

In Chapter 5, we consider the power equations, i.e., the equations of the form

$$dX_t = \mu|X_t|^\alpha dt + \nu|X_t|^\beta dB_t$$

and propose a simple procedure to determine the type of zero and the type of infinity for these SDEs (see Figure 5.1 on p. 94 and Figure 5.2 on p. 98). Moreover, we study which types of zero and which types of infinity are possible for the SDEs with a constant-sign drift (see Table 5.1 on p. 101 and Table 5.2 on p. 103).

The known results from the stochastic calculus used in the proofs are contained in Appendix A, while the auxiliary lemmas are given in Appendix B.

The monograph includes 7 figures with simulated paths of solutions of singular SDEs.

1 Stochastic Differential Equations

In this chapter, we consider general multidimensional SDEs of the form (1.1) given below.

In Section 1.1, we give the standard definitions of various types of the existence and the uniqueness of solutions as well as some general theorems that show the relationship between various properties.

Section 1.2 contains some classical sufficient conditions for various types of existence and uniqueness.

In Section 1.3, we present several important examples that illustrate various combinations of the existence and the uniqueness of solutions. Most of these examples (but not all) are well known. We also find all the possible combinations of existence and uniqueness.

Section 1.4 includes the definition of a martingale problem. We also recall the relationship between the martingale problems and the SDEs.

In Section 1.5, we define a solution up to a random time.

1.1 General Definitions

Here we will consider a general type of SDEs, i.e., multidimensional SDEs with coefficients that depend on the past. These are the equations of the form

$$dX_t^i = b_t^i(X)dt + \sum_{j=1}^{m} \sigma_t^{ij}(X)dB_t^j, \quad X_0 = x_0 \quad (i = 1, \dots, n), \tag{1.1}$$

where $n \in \mathbb{N}$, $m \in \mathbb{N}$, $x_0 \in \mathbb{R}^n$, and

$$b : C(\mathbb{R}_+, \mathbb{R}^n) \times \mathbb{R}_+ \to \mathbb{R}^n,$$
$$\sigma : C(\mathbb{R}_+, \mathbb{R}^n) \times \mathbb{R}_+ \to \mathbb{R}^{n \times m}$$

are predictable functionals. (The definition of a predictable process can be found, for example, in [27, Ch. I, §2 a] or [38, Ch. IV, § 5].)

Remark. We fix a starting point x_0 together with b and σ. In our terminology, SDEs with the same b and σ and with different starting points are different SDEs.

Definition 1.1. **(i)** A *solution* of (1.1) is a pair (Z, B) of adapted processes on a filtered probability space $(\Omega, \mathcal{G}, (\mathcal{G}_t)_{t\ge 0}, \mathsf{Q})$ such that

(a) B is a m-dimensional $(\mathcal{G}_t)$-Brownian motion, i.e., B is a m-dimensional Brownian motion started at zero and is a $(\mathcal{G}_t, \mathsf{Q})$-martingale;

(b) for any $t \ge 0$,

$$\int_0^t \Big(\sum_{i=1}^n |b_s^i(Z)| + \sum_{i=1}^n \sum_{j=1}^m (\sigma_s^{ij}(Z))^2 \Big) ds < \infty \quad \mathsf{Q}\text{-a.s.};$$

(c) for any $t \ge 0$, $i = 1, \dots, n$,

$$Z_t^i = x_0^i + \int_0^t b_s^i(Z) ds + \sum_{j=1}^m \int_0^t \sigma_s^{ij}(Z) dB_s^j \quad \mathsf{Q}\text{-a.s.}$$

(ii) There is *weak existence* for (1.1) if there exists a solution of (1.1) on some filtered probability space.

Definition 1.2. **(i)** A solution (Z, B) is called a *strong solution* if Z is $(\overline{\mathcal{F}}_t^B)$-adapted, where $\overline{\mathcal{F}}_t^B$ is the σ-field generated by $\sigma(B_s;\ s \le t)$ and by the subsets of the Q-null sets from $\sigma(B_s;\ s \ge 0)$.

(ii) There is *strong existence* for (1.1) if there exists a strong solution of (1.1) on some filtered probability space.

Remark. Solutions in the sense of Definition 1.1 are sometimes called *weak solutions*. Here we call them simply *solutions*. However, the existence of a solution is denoted by the term *weak existence* in order to stress the difference between weak existence and *strong existence* (i.e., the existence of a strong solution).

Definition 1.3. There is *uniqueness in law* for (1.1) if for any solutions (Z, B) and $(\widetilde{Z}, \widetilde{B})$ (that may be defined on different filtered probability spaces), one has $\text{Law}(Z_t;\ t \ge 0) = \text{Law}(\widetilde{Z}_t;\ t \ge 0)$.

Definition 1.4. There is *pathwise uniqueness* for (1.1) if for any solutions (Z, B) and $(\widetilde{Z}, B)$ (that are defined on the same filtered probability space), one has $\mathsf{Q}\{\forall t \ge 0,\ Z_t = \widetilde{Z}_t\} = 1$.

Remark. If there exists no solution of (1.1), then there are both uniqueness in law and pathwise uniqueness.

The following 4 statements clarify the relationship between various properties.

Proposition 1.5. *Let* (Z, B) *be a strong solution of* (1.1).

(i) *There exists a measurable map*

$$\Psi : \big(C(\mathbb{R}_+, \mathbb{R}^m), \mathcal{B}\big) \longrightarrow \big(C(\mathbb{R}_+, \mathbb{R}^n), \mathcal{B}\big)$$

(here $\mathcal{B}$ denotes the Borel σ-field) such that the process $\Psi(B)$ is $\big(\overline{\mathcal{F}}_t^B\big)$-adapted and $Z = \Psi(B)$ Q-a.s.

(ii) *If $\widetilde{B}$ is a m-dimensional $(\widetilde{\mathcal{F}}_t)$-Brownian motion on a filtered probability space $\big(\widetilde{\Omega}, \widetilde{\mathcal{G}}, (\widetilde{\mathcal{G}}_t), \widetilde{\mathsf{Q}}\big)$ and $\widetilde{Z} := \Psi(\widetilde{B})$, then $(\widetilde{Z}, \widetilde{B})$ is a strong solution of* (1.1).

For the proof, see, for example, [5].

Now we state a well known result of Yamada and Watanabe.

Proposition 1.6 (Yamada, Watanabe). *Suppose that pathwise uniqueness holds for* (1.1).

(i) *Uniqueness in law holds for* (1.1);

(ii) *There exists a measurable map*

$$\Psi : \big(C(\mathbb{R}_+, \mathbb{R}^m), \mathcal{B}\big) \longrightarrow \big(C(\mathbb{R}_+, \mathbb{R}^n), \mathcal{B}\big)$$

such that the process $\Psi(B)$ is $\big(\overline{\mathcal{F}}_t^B\big)$-adapted and, for any solution (Z, B) of (1.1), *we have $Z = \Psi(B)$ Q-a.s.*

For the proof, see [48] or [38, Ch. IX, Th. 1.7].

The following result complements the theorem of Yamada and Watanabe.

Proposition 1.7. *Suppose that uniqueness in law holds for* (1.1) *and there exists a strong solution. Then pathwise uniqueness holds for* (1.1).

This theorem was proved by Engelbert [10] under some additional assumptions. It was proved with no additional assumptions by Cherny [7].

The crucial fact needed to prove Proposition 1.7 is the following result. It shows that uniqueness in law implies a seemingly stronger property.

Proposition 1.8. *Suppose that uniqueness in law holds for* (1.1). *Then, for any solutions (Z, B) and $(\widetilde{Z}, \widetilde{B})$ (that may be defined on different filtered probability spaces), one has* $\operatorname{Law}(Z_t, B_t;\, t \ge 0) = \operatorname{Law}(\widetilde{Z}_t, \widetilde{B}_t;\, t \ge 0)$.

For the proof, see [7].

The situation with solutions of SDEs can now be described as follows.

It may happen that there exists no solution of (1.1) on any filtered probability space (see Examples 1.16, 1.17).

It may also happen that on some filtered probability space there exists a solution (or there are even several solutions with the same Brownian motion), while on some other filtered probability space with a Brownian motion there exists no solution (see Examples 1.18, 1.19, 1.20, and 1.24).

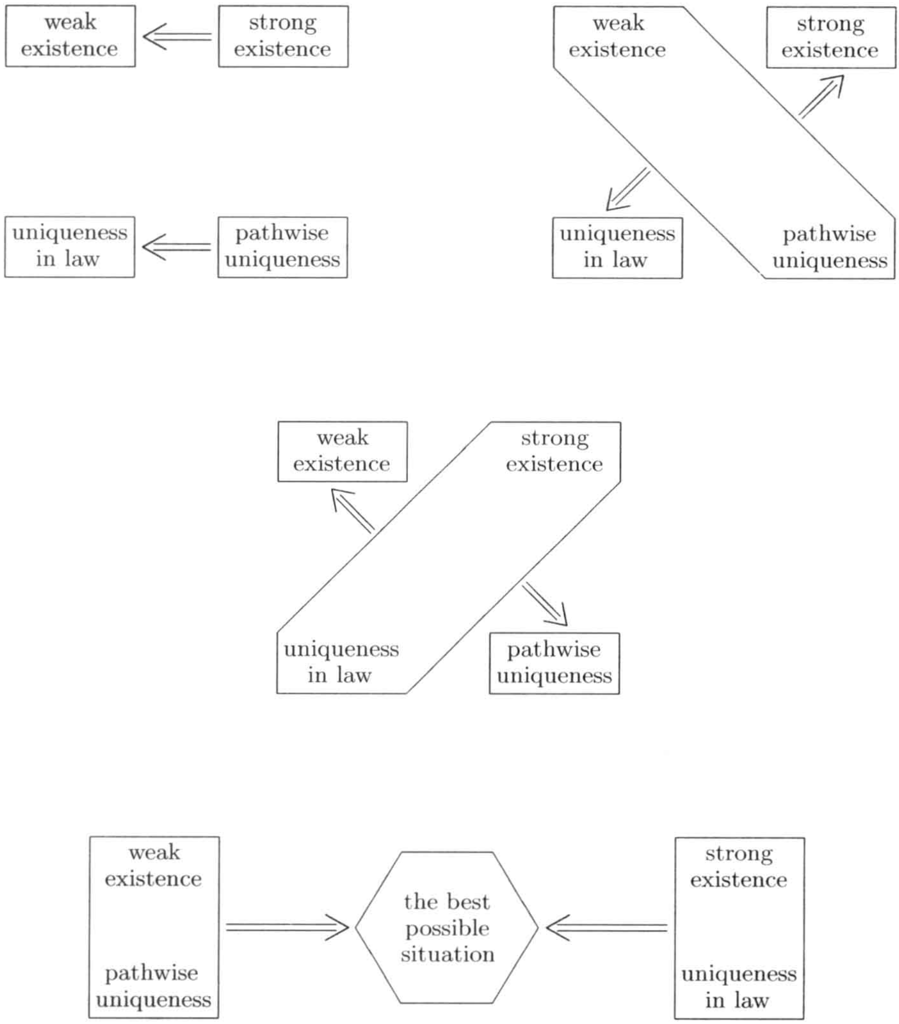

Fig. 1.1. The relationship between various types of existence and uniqueness. The top diagrams show obvious implications and the implications given by the Yamada–Watanabe theorem. The centre diagram shows an obvious implication and the implication given by Proposition 1.7. The bottom diagram illustrates the Yamada–Watanabe theorem and Proposition 1.7 in terms of the "best possible situation".

If there exists a strong solution of (1.1) on some filtered probability space, then there exists a strong solution on any other filtered probability space with a Brownian motion (see Proposition 1.5). However, it may happen in this case that there are several solutions with the same Brownian motion (see Examples 1.21–1.23).

If pathwise uniqueness holds for (1.1) and there exists a solution on some filtered probability space, then on any other filtered probability space with a Brownian motion there exists exactly one solution, and this solution is strong (see the Yamada–Watanabe theorem). This is the best possible situation.

Thus, the Yamada–Watanabe theorem shows that pathwise uniqueness together with weak existence guarantee that the situation is the best possible. Proposition 1.7 shows that uniqueness in law together with strong existence guarantee that the situation is the best possible.

1.2 Sufficient Conditions for Existence and Uniqueness

The statements given in this section are related to SDEs, for which $b_t(X) = b(t, X_t)$ and $\sigma_t(X) = \sigma(t, X_t)$, where $b : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}^n$ and $\sigma : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}^{n\times m}$ are measurable functions.

We begin with sufficient conditions for strong existence and pathwise uniqueness. The first result of this type was obtained by Itô.

Proposition 1.9 (Itô). *Suppose that, for a SDE*

$$dX_t^i = b^i(t, X_t)dt + \sum_{j=1}^m \sigma^{ij}(t, X_t)dB_t^j, \quad X_0 = x_0 \quad (i = 1, \dots, n),$$

there exists a constant $C > 0$ such that

$$\begin{aligned} \|b(t,x) - b(t,y)\| + \|\sigma(t,x) - \sigma(t,y)\| &\le C\|x - y\|, \quad t \ge 0,\ x, y \in \mathbb{R}^n, \\ \|b(t,x)\| + \|\sigma(t,x)\| &\le C(1 + \|x\|), \quad t \ge 0,\ x \in \mathbb{R}^n, \end{aligned}$$

where

$$\begin{aligned} \|b(t,x)\| &:= \left(\sum_{i=1}^n (b^i(t,x))^2\right)^{1/2}, \\ \|\sigma(t,x)\| &:= \left(\sum_{i=1}^n \sum_{j=1}^m (\sigma^{ij}(t,x))^2\right)^{1/2}. \end{aligned}$$

Then strong existence and pathwise uniqueness hold.

For the proof, see [25], [29, Ch. 5, Th. 2.9], or [36, Th. 5.2.1].

Proposition 1.10 (Zvonkin). *Suppose that, for a one-dimensional SDE*

$$dX_t = b(t, X_t)dt + \sigma(t, X_t)dB_t, \quad X_0 = x_0,$$

the coefficient b is measurable and bounded, the coefficient σ is continuous and bounded, and there exist constants $C > 0$, $\varepsilon > 0$ such that

$$|\sigma(t,x) - \sigma(t,y)| \le C\sqrt{|x-y|}, \quad t \ge 0,\ x, y \in \mathbb{R},$$
$$|\sigma(t,x)| \ge \varepsilon, \quad t \ge 0,\ x \in \mathbb{R}.$$

Then strong existence and pathwise uniqueness hold.

For the proof, see [49].

For homogeneous SDEs, there exists a stronger result.

Proposition 1.11 (Engelbert, Schmidt). *Suppose that, for a one-dimensional SDE*

$$dX_t = b(X_t)dt + \sigma(X_t)dB_t, \quad X_0 = x_0,$$

$\sigma \ne 0$ at each point, $b/\sigma^2 \in L^1_{\text{loc}}(\mathbb{R})$, and there exists a constant $C > 0$ such that

$$|\sigma(x) - \sigma(y)| \le C\sqrt{|x-y|}, \quad x, y \in \mathbb{R},$$
$$|b(x)| + |\sigma(x)| \le C(1 + |x|), \quad x \in \mathbb{R}.$$

Then strong existence and pathwise uniqueness hold.

For the proof, see [15, Th. 5.53].

The following proposition guarantees only pathwise uniqueness. Its main difference from Proposition 1.10 is that the diffusion coefficient here need not be bounded away from zero.

Proposition 1.12 (Yamada, Watanabe). *Suppose that, for a one-dimensional SDE*

$$dX_t = b(t, X_t)dt + \sigma(t, X_t)dB_t, \quad X_0 = x_0,$$

there exist a constant $C > 0$ and a strictly increasing function $h : \mathbb{R}_+ \to \mathbb{R}_+$ with $\int_0^{0+} h^{-2}(x)dx = +\infty$ such that

$$|b(t,x) - b(t,y)| \le C|x-y|, \quad t \ge 0,\ x, y \in \mathbb{R},$$
$$|\sigma(t,x) - \sigma(t,y)| \le h(|x-y|), \quad t \ge 0,\ x, y \in \mathbb{R}.$$

Then pathwise uniqueness holds.

For the proof, see [29, Ch. 5, Prop. 2.13], [38, Ch. IX, Th. 3.5], or [39, Ch. V, Th. 40.1].

We now turn to results related to weak existence and uniqueness in law. The first of these results guarantees only weak existence; it is almost covered by further results, but not completely. Namely, here the diffusion matrix σ need not be elliptic (it might even be not a square matrix).

Proposition 1.13 (Skorokhod). *Suppose that, for a SDE*

$$dX_t^i = b^i(t, X_t)dt + \sum_{j=1}^{m} \sigma^{ij}(t, X_t)dB_t^j, \quad X_0^i = x_0^i \quad (i = 1, \dots, n),$$

the coefficients b and σ are continuous and bounded. Then weak existence holds.

For the proof, see [42] or [39, Ch. V, Th. 23.5].

Remark. The conditions of Proposition 1.13 guarantee neither strong existence (see Example 1.19) nor uniqueness in law (see Example 1.22).

In the next result, the conditions on b and σ are essentially relaxed as compared with the previous proposition.

Proposition 1.14 (Stroock, Varadhan). *Suppose that, for a SDE*

$$dX_t^i = b^i(t, X_t)dt + \sum_{j=1}^{n} \sigma^{ij}(t, X_t)dB_t^j, \quad X_0 = x_0 \quad (i = 1, \dots, n),$$

the coefficient b is measurable and bounded, the coefficient σ is continuous and bounded, and, for any $t \geq 0$, $x \in \mathbb{R}^n$, there exists a constant $\varepsilon(t,x) > 0$ such that

$$\|\sigma(t,x)\lambda\| \geq \varepsilon(t,x)\|\lambda\|, \quad \lambda \in \mathbb{R}^n.$$

Then weak existence and uniqueness in law hold.

For the proof, see [44, Th. 4.2, 5.6].

In the next result, the diffusion coefficient σ need not be continuous. However, the statement deals with homogeneous SDEs only.

Proposition 1.15 (Krylov). *Suppose that, for a SDE*

$$dX_t^i = b^i(X_t)dt + \sum_{j=1}^{n} \sigma^{ij}(X_t)dB_t^j, \quad X_0 = x_0 \quad (i = 1, \dots, n),$$

the coefficient b is measurable and bounded, the coefficient σ is measurable and bounded, and there exist a constant $\varepsilon > 0$ such that

$$\|\sigma(x)\lambda\| \geq \varepsilon\|\lambda\|, \quad x \in \mathbb{R}^n, \ \lambda \in \mathbb{R}^n.$$

Then weak existence holds. If moreover $n \leq 2$, then uniqueness in law holds.