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Edited by A. Dold, B. Eckmann and F. Takens

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Global Solution Branches of Two Point Boundary Value Problems

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Introduction

For the parameter dependent problem

$$\text{(I-1-1)} \qquad \begin{aligned} &u''(x) + \lambda^2 f(u(x)) = 0 \quad , \quad \lambda > 0 \\ &u(0) = u(1) = 0 \end{aligned}$$

the following is known from application of general local ([13]) or global ([24]) bifurcation theorems in the case $f(0) = 0$, $f'(0) > 0$:

The trivial solution $u \equiv 0$ exists for each λ. From this trivial solution branch there is bifurcation of nontrivial solutions in each point $(0, \lambda)$ for which the linearized problem (I-1-1) has a nontrivial kernel, i.e., for $\lambda = i\pi/\sqrt{f'(0)}$, $i = 1, 2, \ldots$. The bifurcating branches, i.e., connected components of nontrivial solutions with bifurcation points in the (λ, u)-space, are all unbounded and each branch consists of solutions (λ, u) where u has a number of simple zeroes in $]0,1[$ which is characteristic for the branch. In the solution branches bifurcating from 0 all u are bounded by the first positive and first negative zero of f.

In applications more information about the shape of solution branches is needed. It is easy to see that all branches are in fact curves which are at least as smooth as f is (see below). It is then of interest whether these curves have turns with respect to the λ-direction or not, and if so, how many turning points there are and where they are located. Let, e.g., the branch of positive and negative solutions to (I-1-1) look like this:

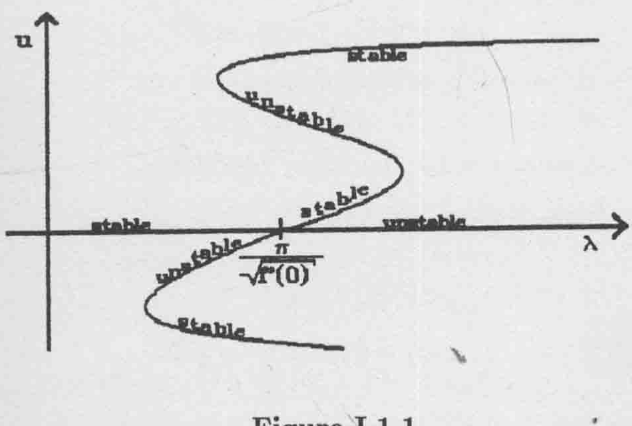

Figure I.1.1

Then (I-1-1) is the stationary equation of

(I-1-2) $$u_t = u_{xx} + \lambda^2 f(u)$$ $$u(t,0) = u(t,1) = 0$$

and the directions of the solution branches determine the stability of the stationary states as indicated.

In combustion problems there often occurs an equation of the form (I-1-1) describing intermediate steady states of the temperature distribution u, λ then measures the amount of unburnt substance. In this context turning points of a branch correspond to ignition and extinction points of the process, and it is of importance whether or not those exist (see e.g.[15]).

In some cases one can skilfully choose sub- and supersolutions of (I-1-1) and then get the result that there exist at least two stable solutions for a certain λ-interval. Then using degree theory there has to be at least one more unstable solution for any λ in this interval. This way it is shown that turning points of the branch have to exist, but one only gets estimates from below for both the number of solutions and the number of turns of the branch. It is not possible to give upper estimates of these numbers without using strong analytical tools.

We use for this purpose the so called time map T (see [31]) of the nonlinearity f:

First of all by a scaling of x, $x = \lambda t$ we can write (I-1-1) in the form

(I-1-3) $$u''(t) + f(u(t)) = 0$$ $$u(0) = u(\lambda) = 0$$

thus having the parameter in the boundary condition.

If $u(t) = U(t,p)$ is the solution of the initial value problem

(I-1-4) $$u'' + f(u) = 0$$ $$u(0) = 0\,,\; u'(0) = p \neq 0$$

then we define $T(p) = T_1(p)$ to be the first positive t for which $U(t,p)$ is zero again, if this one exists for the given p:

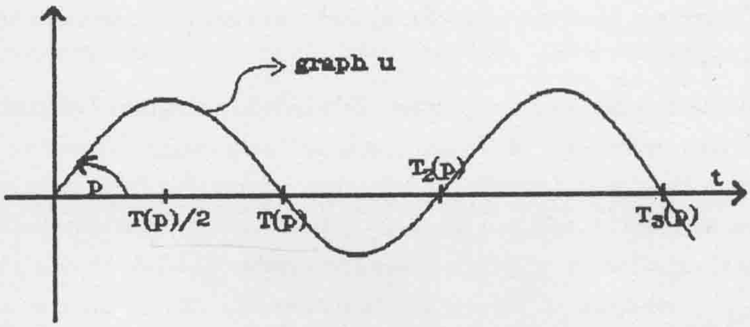

Figure I.1.2

$T_2(p)$ is the second zero of u, $T_3(p)$ the third (existence provided) etc..

Then the solution branches of (I-1-3) are given by the graphs of the T_i :

(λ, u) solves (I-1-3) if and only if $u = U(\cdot, p)$, $T_i(p)$ is defined for some $i = 1, 2, ...$ and

(I-1-5) $$\lambda = T_i(p) .$$

Stable regions of positive or negative solution branches are $u'(0) = p$-regions with $pT'(p) > 0$, unstable ones correspond to $pT'(p) < 0$, turning points of branches correspond to $T'(p_0) = 0$, $T''(p_0) \neq 0$.

Time maps and period maps (see below) have been studied in several papers, see [3], [5], [8], [9], [22], [25], [26], [31], [33], [35], [36], a list of references which is by no means complete.

By the implicit function theorem T_i is always at least as regular as f is since

$$U(T_i(p), p) = 0 \qquad \left|\frac{\partial}{\partial t} U(T_i(p), p)\right| = |p| > 0 .$$

The last identity follows since (I-1-4) has a first integral

(I-1-6) $$\frac{1}{2}(u')^2 + F(u) \equiv const = \frac{1}{2}p^2$$

for $u = U(\cdot, p)$ and F being the integral of f with $F(0) = 0$.

Note that with T and T_i we get all solutions of (I-1-3), not only the branches bifurcating from 0. Also it is not necessary that we have a trivial solution set via $f(0) = 0$. (There are general results for global branches in these cases too, we did not mention them.)

From (I-1-6) one can derive a formula for T which is a singular integral (see (1-1-3)). For studying derivatives of T this one is not very useful. Therefore in section 1.1 we derive a time map formula for branches bifurcating from zero which does not contain a singularity and can be easily differentiated. The idea behind this is to use a transformation in u which maps the orbits of (I-1-4) in the phase plane into circles. For solutions in bifurcating branches this can be done since f does not change sign more than once along the range of such solutions.

Chapter 1 consists entirely of applications of this time map formula. Section 1.2 recalls results about bifurcation points and bifurcation directions. In 1.3 we use the time map formula to reprove results by Chafee, Infante ([5]) and Opial ([22]) giving conditions for branches which have only a single turn at the bifurcation point itself.

Paragraph 1.4 goes a step further and gives conditions on f under which T_i'' does not change sign. This results in branches which still have at most one turning point, but this one can occur away from the bifurcation point (see figure I.1.2). This allows for a change of stability along branches, but stability can change at most twice.

We have tried to find a condition on f for $T_i'' \neq 0$ which can be verified in examples. For this we introduce the notion of an A-B-function which is a function f with

$$
\text{(I-1-7)} \qquad \begin{aligned} f'f''' - \frac{5}{3}(f'')^2 &< 0 \quad \text{in regions where } f' \geq 0. \\ ff'' - 3(f')^2 &< 0 \quad \text{in regions where } f' \leq 0. \end{aligned}
$$

It is shown that $T_i'' > 0$ for all bifurcating branches of A-B-functions. This class contains all polynomials without complex zeroes as is shown in 1.5. With this we have generalized the result of Smoller and Wasserman in [31] which says that the T_i have at most one critical point which then is a minimum in the case that f is a cubic $f(u) = -u(u-\alpha)(u-\beta)$ with $\alpha < 0 < \beta$. 1.5 contains more methods for getting hold of A-B-functions, using, e.g., the Schwarzian of f. See also the examples discussed near the end of this introduction.

Section 1.6 deals with the asymptotic behaviour of T as p approaches a boundary point of its definition set. Some of the results can be used to prove existence of dead core solutions to certain reaction diffusion problems (see examples 1.6.2 and 3.2.2).

Chapter 2 contains results on the Neumann problem corresponding to (I-1-3)

$$\text{(I-1-8)} \qquad \begin{aligned} &u'' + f(u) = 0 \\ &u'(0) = u'(\lambda) = 0\,, \end{aligned}$$

and related results on period maps of Hamiltonian systems

$$\text{(I-1-9)} \qquad u' = f_2(v) \qquad v' = -f_1(u)\,.$$

If r is some zero of f then $u \equiv r$ is always a trivial solution of (I-1-8). The linearization of (I-1-8) about $u \equiv r$ has a nonzero solution if and only if $f'(r) > 0$, $\lambda = i\pi/\sqrt{f'(r)}$, $i = 1, 2, \dots$. These are the bifurcation points of (I-1-8). Solutions u in the bifurcating branches are bounded by the first zeroes of f below and above r. For these branches one can show that they have a single turn at the bifurcation point in the case that f is an A-B-function. The proof uses a Neumann time map defined analogously to the Dirichlet time map mentioned before.

With about the same method we get monotonicity results for the period map of (I-1-9), which is the map assigning to E the least period $\Pi(E)$ of a periodic solution of (I-1-9) with energy $E \equiv F_1(u) + F_2(v)$. Such results can be used to prove bifurcation of subharmonic solutions if (I-1-9) is perturbed by a small nonautonomous t-periodic term (see [7]). Another application of such results is the existence and uniqueness of period-4-solutions to the time delay equation $u'(t) + f(u(t-1)) = 0$ with antisymmetric f (see [18]).

In chapter 3 we return to the Dirichlet problem and give results for problem (I-1-3) where we assume less for f and also consider non bifurcating branches:

In 3.1 we no longer assume $f(0) = 0$, $f'(0) > 0$ but just consider some f defined on $]0, a[$ which is positive there. This way we can handle the branch of (I-1-3) which consists of solutions (λ, u) with u positive and u'' not changing sign. For these we get a time map formula similar to the one in 1.1.

The regularity of f required for the existence of T turns out to be a fairly weak one. Therefore problems with singularities at $u = 0$ can be dealt with. Multiple zeroes of f at 0 can also occur. In this case there is "bifurcation" from $(\lambda = \infty, u \equiv 0)$. If f is an A-B-function then the branch is shown to be again U-shaped.

The case $f(0) > 0$ shows more difficulties than $f(0) = 0$: For $f(u) = -(u - \alpha)(u - \beta)(u - \gamma)$ with $\alpha < \beta < 0 < \gamma$ it is shown in [31] that the number of critical points of T is at most two, and exactly two if β is close to 0. So far we are not able to generalize this to A-B-functions f, though it seems like this result is true. The only thing we could do is to give a criterion on f such that T has at most one critical point. In the case $f(0) > 0$ it is not possible to show this via a sign of T'' or a related expression. Instead we use variation diminishing properties of integral operators proved in [19], results are to be found in 3.3.

In 3.4 we discuss positive solution branches in which u'' changes sign, i.e. u has at least one zero of f in its range. Under certain assumptions on f we can again show that branches are U-shaped.

The last chapter discusses some general features of Dirichlet time maps. In 4.1 we give proofs for the relationship between the sign of $pT'(p)$ and stability. Also there is a strong connection between $T(p)$ and the energy level of the corresponding solution of (I-1-3) if we define the energy level via the Liapounov functional

$$E(u) = \int_0^\lambda \frac{1}{2}(u'(t))^2 - F(u(t))\,dt\,.$$

It turns out that E is decreasing along parts of solution branches with $pT'(p) > 0$ and increasing if $pT'(p) < 0$. As another result if there are multiple solutions for a fixed λ an unstable solution always has higher energy than the two "nearest" stable solutions.

In section 4.2 we again only consider nonlinearities f defined on some interval $]0, a[$ which are positive there. We pose the inverse problem if it is possible to decide for a given curve $p \mapsto T(p)$ whether or not it is the time map of some f and if it is possible to calculate f from T, i.e., to get the problem back from its solution branch. The set of all time maps of such f is characterized by an integral condition and the operator $f \mapsto T$ is shown to be invertible.

Applications include proofs of generic properties of such time maps analogous to [3], [33], as well as estimates for λ-regions of existence and nonexistence for (I-1-3), and can be found in 4.3 among some games about possible and impossible time maps.In the appendix, finally, we briefly discuss the method to obtain time map plots via the computer.

In order to give a flavour of the possible applications results in this monograph can have let us discuss two specific examples. Please note that the argumentation used to interprete results is not rigorous up to the detail.

It is a good illustration to view (I-1-1) as the stationary equation for a population of size u diffusing on the interval $[0,1]$ which has a hostile environment thus forcing $u(t,0) = u(t,1) = 0$. $1/\lambda^2$ can then be regarded to be the diffusion coefficient of the population, wheras $f^+(u) = f(u)/u$ models the reproduction rate consisting of the birth rate minus the death rate.

Let us first consider the example

$$f^+(u) = e^{-(u-\alpha)^2} \quad , \quad \alpha \geq 0 .$$

This means that the reproduction rate is maximal for a certain optimal population size α and declines to 0 as u becomes large but the death rate never exceeds the birth rate. This means for the model without diffusion

$$u_t = uf^+(u)$$

that any population with a positive intitial size grows unlimited. If we now add diffusion

$$\text{(I-1-10)} \qquad \begin{aligned} u_t &= \frac{1}{\lambda^2}u_{xx} + uf^+(u) \\ u(t,0) &= u(t,1) = 0 \end{aligned}$$

then the behaviour depends on the size of λ and is in principal governed by the bifurcation diagram for the stationary equation (I-1-1).

All stationary nontrivial branches are given by $\lambda = T_i(p)$. In this specific context only the positive solution branch $\lambda = T(p)$, $p > 0$, matters since the system is not going to get near any sign changing steady state if it starts out with $u(0,x) \geq 0$ (maximum principle). Section 1.2 gives the bifurcation points $(\lambda_i, u \equiv 0)$ as

$$\lambda_i = T_i(0) = \frac{i\pi}{\sqrt{f'(0)}} = i\pi\sqrt{e^{\alpha^2}}$$

and the bifurcation directions as

$$T_i'(0) = \begin{cases} 0, & i \text{ even} \\ -\dfrac{2}{3}\dfrac{f''}{(f')^2}(0) = -\dfrac{2}{3}\alpha e^{\alpha^2}, & i \text{ odd} \end{cases} .$$

So the positive branch starts out unstable if $\alpha > 0$.

For the "other end" of the branches we use section 1.6: First of all the definition set of the T_i is (see section 1.1) $D(T_1) =]b^-, b^+[$ with $b^- = -\sqrt{2F(-\infty)}$, $b^+ = \sqrt{2F(\infty)}$ and $D(T_i) =]b^-, -b^-[$ for $i \geq 2$.

From proposition 1.6.1 (iv) it follows that $\lim_{p\to b+} T(p) = \frac{\pi}{\sqrt{c^+}}$ with $c^+ = \lim_{u\to\infty} \frac{f^2(u)}{2F(u)} = \lim_{u\to\infty} f^+(u) = 0$ and $\lim_{p\to b-} T(p) = \frac{\pi}{\sqrt{c^-}}$ with $c^- = \lim_{u\to-\infty} f^+(u) = 0$. So

$$\lim_{p\to b^+, b^-} T(p) = +\infty \,.$$

Formulas (1-6-17) and (1-6-18) then imply for all the other branches that $\lim_{p\to b^-,-b^-} T_i(p) = +\infty$, $i \geq 2$.

Let us put together what we have found out by this for the positive branch $\lambda = T(p)$, $p > 0$: It starts out unstable but has to recover stability later since $T(p) \to +\infty$ as $p \to b^+$. If we let $\lambda^* = \min_{p\geq 0} T(p) > 0$ then no nontrivial steady state exists for $\lambda < \lambda^*$ and the zero solution is a global attractor for the system ([14], [17]). The population dies out from too fast diffusion and the hostile environment. For $\lambda > \lambda_1 = \pi e^{\alpha^2/2}$ the zero solution looses its stability, but the population could never grow beyond limits: From the Liapounov functional and the fact that $f^+(+\infty) = 0$ it follows that any solution of (I-1-10) is a priori bounded. So diffusion and the zero boundary conditions prevent unlimited growth.

Results in [17] can be used to obtain further information on the dynamical behaviour of our system for $\lambda > \lambda^*$ if we know that all steady states are hyperbolic (this follows from $T_i'(p) \neq 0$ if $\lambda = T_i(p)$) and if we know more about the maximal invariant set A which consists of the unstable manifolds of all steady states present for our λ. For this example this is possible by applying sections 1.3, 1.4, 1.5 and the results about connecting orbits obtained by Brunovský and Fiedler in [4].

Let us first consider the case $\alpha = 0$. Then $u\frac{d}{du}\frac{f(u)}{u} = uf^{+\prime}(u) < 0$ and we can use theorem 1.3.2 to obtain that $pT_i'(p) > 0$, which completely proves the following bifurcation diagram:

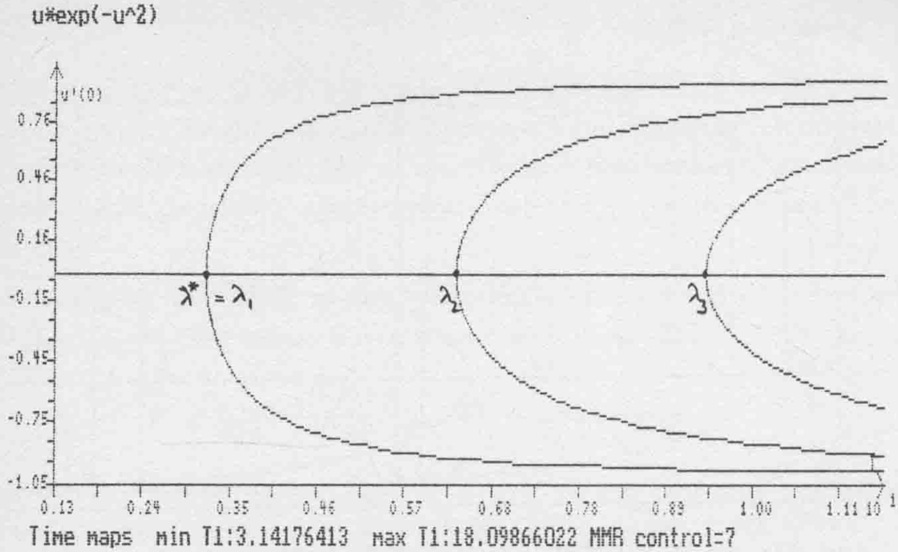

Figure I.1.3

In this case λ coincides with λ^* and it follows ([14], [17]) that our population dies out for $\lambda < \lambda_1$ and for $\lambda > \lambda_1$ is attracted by the single stable positive steady state represented by $\lambda = T(p)$.

If we do not assume that the maximal reproduction rate occurs at $u = 0$, i.e., if we take $\alpha > 0$, then we know already that $T'(p) < 0$ near $p = 0$. So there is at least one turning point of the positive branch in the nontrivial range. We can show that there is at most one by showing that $T'' > 0$. This again follows from the results in 1.4 if f is an A-B-function (see (I-1-7)). Now it is clear that (I-1-7) follows from $(\ln|f|)'' < 0$ and $(\ln|f'|)'' < 0$ on regions where these are defined and from $ff''(u) < 0$ whenever $f'(u) = 0$. The last and the first condition can be easily checked. For the middle one we notice that

$$f'(u) = e^{-(u-\alpha)^2}(A-u)(u-B)$$

with $A = \frac{1}{2}(\alpha + \sqrt{\alpha^2+2})$, $B = \frac{1}{2}(\alpha - \sqrt{\alpha^2+2})$, thus

$$(\ln|f'|)''(u) = -2 - \frac{1}{(A-u)^2} - \frac{1}{(u-B)^2} < 0.$$

So we have proved the following bifurcation picture:

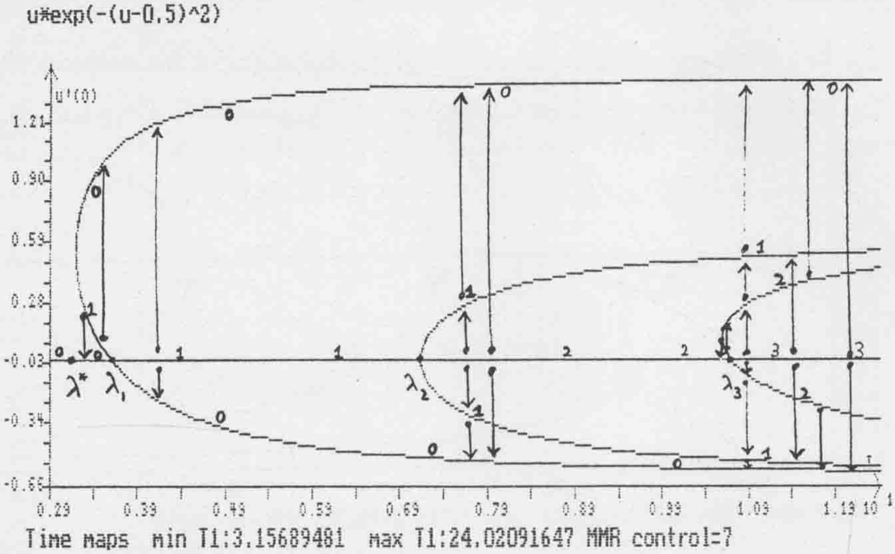

Figure I.1.4

This diagram determines the complete qualitative dynamical behaviour of our time dependent system, also if we additionally consider initial conditions which are allowed to be negative in subsets of $]0,1[$: The numbers along the branches indicate the dimension of the unstable manifold of each of the stationary solutions in that part of the branch. In [4] it is proved that for this type of a global bifurcation diagram there is an orbit of (I-1-10) from any given unstable steady state u to each steady state v whoose unstable dimension is lower. These connecting orbits are indicated as arrows. The dimension of the set of orbits connecting u to v is just the difference between their unstable dimensions. Stationary points together with their connecting orbits form the maximal compact invariant set A for the flow to which everything is eventually attracted ([17]). Since we can deduce the flow on A from the bifurcation diagram, the complete qualitative behaviour of (I-1-10) is known. Every initial distribution which does not lie in either one of the stable manifolds of the unstable solutions will be attracted to one of the stable steady states. This set of initial conditions is open and dense in $H^1([0,1])$. If we only consider initial conditions $u(0,x) \geq 0$, $\not\equiv 0$, as is appropriate in our model context then the following can be said: For $\lambda < \lambda^*$ we always have $\lim_{t\to\infty} u(t,x) = 0$. If $\lambda^* < \lambda < \lambda_1$ then there are two positive steady states $u_1 < u_2$ with u_1 being unstable, u_2 stable and with $u \equiv 0$ being an additional stable steady state. Then the unstable manifold of u_1 (a set of codimension 1) divides the set of initial conditions into two components which are then either attracted by u_2 or 0. If $u(0,x) > u_1(x)$ then $\lim_{t\to\infty} u(t,x) = u_2(x)$ whereas for $u(0,x) < u_1(x)$ we get $\lim_{t\to\infty} u(t,x) = 0$, the initial population size was not large enough to allow survival. For $\lambda > \lambda_1$ the zero solution has become unstable and there is a single positive steady state to which any positive initial condition is attracted.

Next it might be interesting to give an estimate of the value λ^* below which everything dies out. We can do this in theory using proposition 4.3.2 in chapter 4:

$$\lambda^* \geq \min_{u\geq 0} \pi \frac{\sqrt{2F(u)}}{f(u)}, \qquad \lambda^* \leq \min_{u\geq 0} \pi \frac{u}{\sqrt{2F(u)}}.$$

But since $F(u)$ cannot be calculated if $\alpha > 0$ we can only give a more crude concrete estimate from below using f': From (4-3-4) we get

$$\min_{u\geq 0} \pi \frac{\sqrt{2F(u)}}{f(u)} \geq \frac{\pi}{\sqrt{\max_{u\geq 0} f'(u)}}.$$

Now $f'(u) = e^{-(u-\alpha)^2}(1+2\alpha u - 2u^2) \leq \max_{u\geq 0}(1+2\alpha u-2u^2) = 1+\alpha^2/2$. So

$$\lambda^* \geq \frac{\pi\sqrt{2}}{\sqrt{2+\alpha^2}}.$$

In the previous example no unlimited growth in the presence of diffusion was possible because of $f^+(\infty) = 0$. This is different if we choose

$$f^+(u) = \frac{u-1}{u+1}.$$

The choice of the production rate carries another difference: It is negative for u near 0, which is an appropriate modelling for a population with sexual reproduction. This way we no longer have $f'(0) > 0$, thus no bifurcation from the trivial solution branch occurs and the the results of section 3.4 for nonbifurcating branches apply. From there we get that the time map for positive solutions $T(p)$ is defined for $p \in \,]0,\infty[$. Representation (3-4-13) for $T(p)$ together with proposition 3.1.4 shows that $\lim_{p\to\infty} T(p) = \pi\sqrt{c}$ with $c = \lim_{u\to\infty} \frac{2F(u)}{f^2(u)} = \frac{1}{f^+(+\infty)} = 1$ and that $\lim_{p\to 0} T(p) = +\infty$.

In the case that f starts out negative near 0 and only has a single sign change on $]0,\infty[$ condition (3-4-23) for theorem 3.4.4 is always satisfied. This tells us that

$$\text{(I-1-11)} \qquad \frac{d^2}{d\hat{p}^2} T(p) > 0 \quad \text{with } p^2 = \hat{p}^2 + 2F(1)$$

provided f is an A-B-function where it is positive, that is on $]1,\infty[$. Since f' is positive there we only have to check the first part of (I-1-7) which follows easily from $f''' < 0$.

So there is a change of variables for T which turns it into a convex function. Because of the asymptotic behaviour we then conclude that T' is always negative and the $\lambda^* = \pi$ is the minimal value of T:

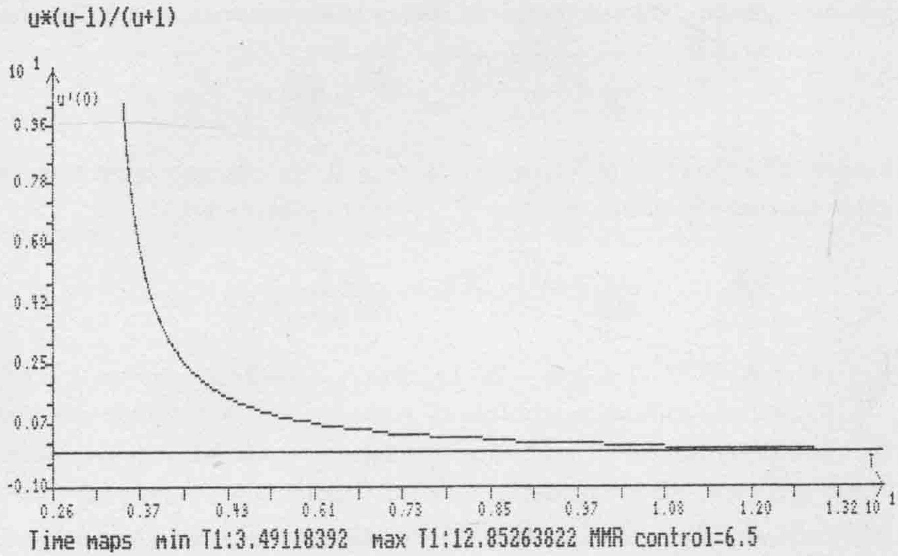

Figure I.1.5

Hence, as in the model without diffusion the $0-$ solution is always stable, but its region of attraction changes with λ. If the unstable steady state u_1 given by $\lambda = T(p)$ is present (for $\lambda > \lambda^*$) then any initial condition below u_1 is attracted to 0 whereas we expect anything above it to grow unlimited.

In this example it is interesting to see the influx of a diffusion rate which depends on the population density. If we modify our model to

$$\text{(I-1-12)} \qquad \begin{aligned} u_t &= \frac{1}{\lambda^2}(d(u)u_x)_x + f(u) \\ u(t,0) &= u(t,1) = 0 \end{aligned}$$

with $d(u) > 0$ then the bifurcation diagram of steady states can change with the form of $d(u)$. With $D(u) := \int_0^u d(s)\,ds$, h being the inverse function of D and with $u = h(v)$ (I-1-12) becomes equivalent to

$$\text{(I-1-13)} \qquad \begin{aligned} (h(v))_t &= \frac{1}{\lambda^2} v_{xx} + f(h(v)) \\ v(t,0) &= v(t,1) = 0 . \end{aligned}$$

This problem has a stationary equation (I-1-1) with v taking the place of u and $\tilde{f} := f \circ h$ the place of f.

Since in our example $f''' < 0$ we not only have that f is an A-B-function but that the Schwarzian Sf of f (see 1.5) is negative where $f, f' > 0$. Then by lemma 1.5.7 we get that $f \circ h$ is an A-B-function for any h with $Sh \leq 0$. This is in particular true if h is a linear fractional transformation because then $Sh = 0$. So choosing any linear fractional transformation for D will give us the result that T is a convex function modulo a transformation in p. The particular behaviour of T then only depends on the aymptoticts near the boundary of its definition set. Let us choose the two possible examples of D which yield two rather different bifurcation diagrams:

First consider

$$D(u) = \frac{\alpha u}{u + \beta} \quad \alpha, \beta > 0\,. \tag{I-1-14}$$

This implies that the diffusion rate $d(u)$ starts at a positve level at $u = 0$ and decreases to 0 as $u \to \infty$, modelling an effect of increasing stickiness. Then $\tilde{f}$ is defined for $0 \leq v < \alpha$, and the definition set of T is again $]0, \infty[$. Since $\tilde{F}(\alpha) = \tilde{f}(\alpha) = +\infty$ we conclude from 3.1.4 together with (3-4-13) that $T(p) \to 0$ as $p \to \infty$. So this is the bifurcation diagram (here $\alpha = 1$, $\beta = 2$):

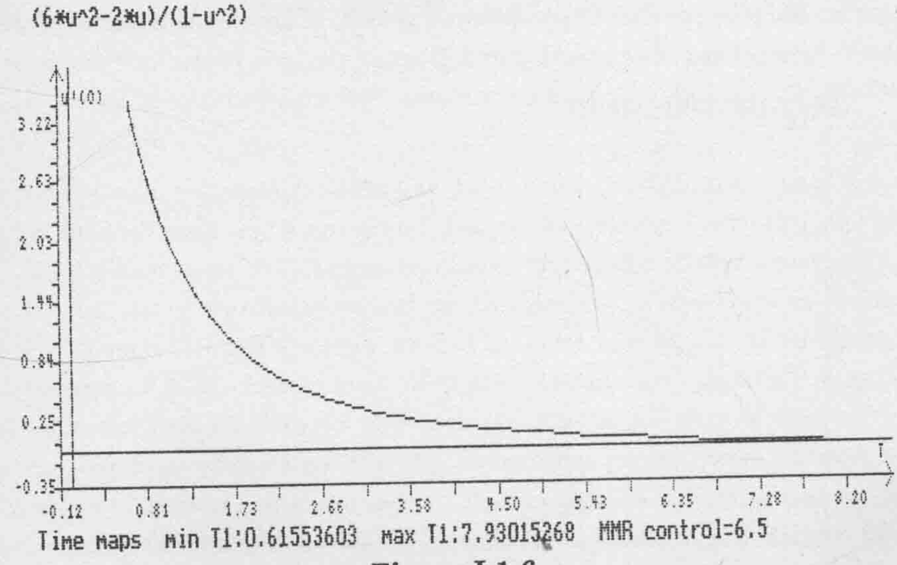

Figure I.1.6

The difference to the situation $d(u) \equiv 1$ is that a "hair trigger" unstable steady state $u_1 = h(v_1)$ is present for all λ. So the strategy of "sticking together" has the effect that the 0 always has a finite region of attraction and that unlimited growth is never excluded. Note that this is possible since the production rate $f^+(u)$ has a positive lower bound as $u \to \infty$ thus we assume infinite resources for the population. There is no way that any choice of $d(u)$ could alter the fact of restricted growth in our first example since there the decrease of f^+ towards 0 is too fast. Actually it is not obvious that the bifurcation diagram here translates into the asymptotic behaviour of (I-1-12) or (I-1-13) in the same way as in the first example. But the reader can easily work out that this is in fact the case using the remarks at the end of section 3.2.

The other possible choice of D is

(I-1-15) $$D(u) = \frac{\alpha u}{\beta - u} \quad , \quad \alpha, \beta > 0 .$$

This way (I-1-12) only makes sense for $0 \leq u < \beta$ and $d(u)$ increases from a positive level at $u = 0$ to infinity as $u \to \beta$. This could model a phobic reaction if the population density becomes too large. (Don't take the model interpretation too serious, it is only included here to make things a little more illustrative. But these are the lines along which "real" models could be discussed.) $\tilde{f}(v) = f(h(v))$ then is defined for $0 \leq v < \infty$ with $\tilde{f}(v)/v \to 0$ as $v \to \infty$ since $h(v) \to \beta$ as $v \to \infty$ and f^+ is bounded. So with the same sort of arguments as before $T(p) \to \infty$ as $p \to \infty$ and $T(p) \to \infty$ as $p \to 0$. This way the following picture is proved (again here $\alpha = 1$ and $\beta = 2$):

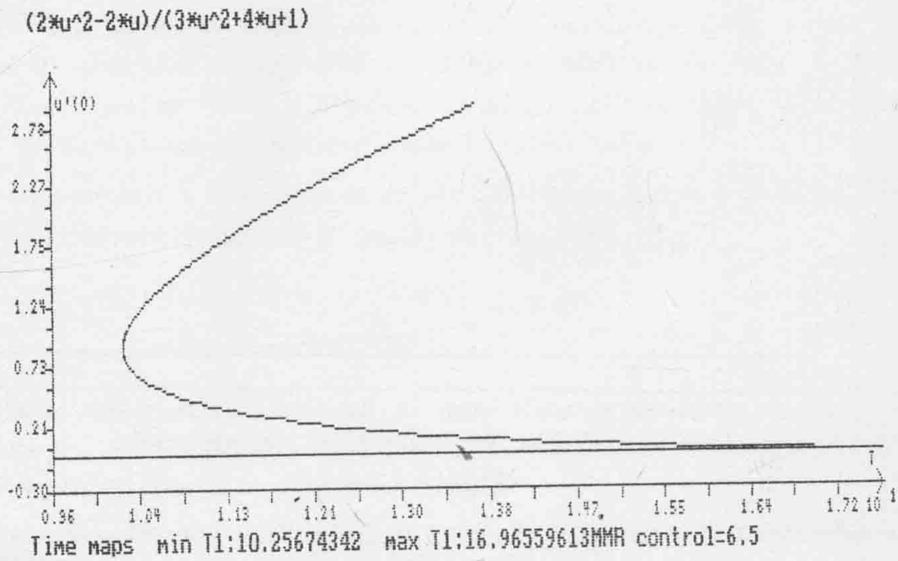

Figure I.1.7