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Wojciech Banaszczyk

Additive Subgroups of Topological Vector Spaces

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Author

Wojciech Banaszczyk
Institute of Mathematics
Łódź University
Banacha 22
90-238 Łódź, Poland

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CONTENTS

Chapter 1

PRELIMINARIES

In this chapter we establish notation and terminology. We also state some standard facts in a form convenient to us. Section 1 is devoted to abelian topological groups and section 2 to topological vector spaces. In section 3 we give some more or less known facts about additive subgroups of R^n.

1. Topological groups

The groups under consideration will be mostly additive subgroups or quotient groups of vector spaces. Therefore we shall apply the additive notation mainly, denoting the neutral element by 0. Naturally, we shall keep the multiplicative notation for groups of, say, non-zero complex numbers or linear operators. The additive groups of integers and of real and complex numbers will be denoted by Z, R and C, respectively. The multiplicative group of complex numbers with modulus 1 will be denoted by S.

By a <u>character</u> of a group G we mean a homomorphism of G into the group $T := R/Z$. We shall frequently identify T with the interval $(-\frac{1}{2},\frac{1}{2}]$. The canonical projection of R onto T will be denoted by ρ. Thus $\rho(x) = x$ for $x = (-\frac{1}{2},\frac{1}{2}]$. The value of character χ at a point g will be denoted by $\chi(g)$ or, sometimes, by $\langle\chi,g\rangle$.

Now, let G be an abelian topological group (we do not assume topological groups to be Hausdorff). The set of all continuous characters of G, with addition defined pointwise, is an abelian group again. We call it the <u>dual group</u> or the <u>character group</u> of G and denote by $G^{\wedge}$.

Characters are usually defined as homomorphisms into S. Such a definition is convenient in harmonic analysis, when we consider complex-valued functions "synthesized" of continuous characters (such a situation will take place in chapter 4). However, it leads to the multiplicative notation on $G^{\wedge}$ which is inconvenient in duality theory when we try to maintain symmetry between G and $G^{\wedge}$ (especially when we consider topological vector spaces). There are also certain technical reasons for which we have chosen T instead of S.

We shall have to consider various topologies on $G^{\wedge}$. The dual group endowed with a given topology τ will be denoted $G^\frown_\tau$. By $G^\frown_p$, $G^\frown_c$ and $G^\frown_{pc}$ we shall denote the dual group endowed, respectively, with the topology of uniform convergence on finite, compact and precompact subsets of G (i.e. with the topology of pointwise, compact and precompact convergence). The second one is usually called the compact-open topology.

Now, let A be a subset of G. If χ is a character of G, then we write

$$|\chi(A)| = \sup \{|\chi(g)|: g \in A\}.$$

The set

$$\{\chi \in G^\frown : |\chi(A)| \leq \frac{1}{4}\}$$

is called the <u>polar</u> of A in $G^\frown$; we denote it by A^o_G. If the meaning of G is clear, we simply write A^o instead of A^o_G. By A^o_p, A^o_c and A^o_{pc} we denote the set A^o endowed with the topology of pointwise, compact and precompact convergence, respectively.

If A is a subgroup of G, then

$$A^o = \{\chi \in G^\frown : \chi|_A \equiv 0\};$$

this follows, for instance, from (1.2). Thus A^o is a closed subgroup of $G^\frown_p$; we call it the <u>annihilator</u> of A.

A subset A of G is said to be <u>quasi-convex</u> if to each $g \in G \setminus A$ there corresponds some $\chi \in A^o$ with $|\chi(g)| > \frac{1}{4}$. The set

$$\bigcap_{\chi \in A^o} \{g \in G : |\chi(g)| \leq \frac{1}{4}\}$$

is evidently the smallest quasi-convex subset of G containing A; we call it the <u>quasi-convex hull</u> of A. We say that G is a <u>locally quasi-convex</u> group if it admits a base at zero consisting of quasi--convex sets. Observe that if G is a Hausdorff locally quasi-convex group, then it admits sufficiently many continuous characters (i.e. continuous characters separate the points of G). Observe also that the polar of any subset of G is a quasi-convex subset of each of the groups $G^\frown_p$, $G^\frown_c$ and $G^\frown_{pc}$; therefore all the three groups are locally quasi-convex.

(1.1) LEMMA. Let g,h be two elements of an abelian group G. If χ is a character of G such that $|\chi(g)|$, $|\chi(h)|$ and $|\chi(g+h)|$ are less than $\frac{1}{3}$, then $\chi(g+h) = \chi(g) + \chi(h)$.

Proof. One has

$$\chi(g + h) \equiv \chi(g) + \chi(h) \quad (\mathrm{mod}\ Z),$$

i.e.

(1) $$\chi(g + h) - \chi(g) - \chi(h) \in Z.$$

From our assumption we obtain

(2) $$|\chi(g + h) - \chi(g) - \chi(h)| \leqq |\chi(g + h)| + |\chi(g)| + |\chi(h)| < \frac{1}{3} + \frac{1}{3} + \frac{1}{3} = 1.$$

Now (1) and (2) imply that $\chi(g + h) - \chi(g) - \chi(h) = 0$. ∎

(1.2) LEMMA. Let χ be a character of an abelian group G. Let m be a positive integer and g an element of G, such that $\chi(kg) < \frac{1}{3}$ for $k = 1,\ldots,m$. Then $\chi(mg) = m\chi(g)$.

Proof. By the preceding lemma, for each $k = 1,\ldots,m-1$, we have $\chi((k + 1)g) = \chi(kg) + \chi(g)$. Thus

$$\sum_{k=1}^{m-1} \chi((k + 1)g) = \sum_{k=1}^{m-1} \chi(kg) + \sum_{k=1}^{m-1} \chi(g),$$

which means that $\chi(mg) = m\chi(g)$. ∎

Let A be a subset of an abelian group G. By $\mathrm{gp}\ A$ we denote the subgroup of G generated by A. For each $m = 1,2,\ldots,$ we denote

$$A^m = \{a_1 + \ldots + a_m : a_1,\ldots,a_m \in A\}.$$

(1.3) PROPOSITION. Let G be an abelian topological group. The polars of compact (resp. finite, precompact) subsets of G form a base at zero in $G_c^{\wedge}$ (resp. in $G_p^{\wedge}$, $G_{pc}^{\wedge}$).

Proof. Let U be a neighbourhood of zero in $G_c^{\wedge}$ (resp. in $G_p^{\wedge}$, $G_{pc}^{\wedge}$). There exist an $\varepsilon > 0$ and a compact (resp. finite, precompact) subset Y of G, such that the set $W = \{\chi \in G^{\wedge} : |\chi(Y)| < \varepsilon\}$ is contained in U. Choose an integer $m > (4\varepsilon)^{-1}$. The set $A = Y^m$ is compact (resp. finite, precompact). By (1.2), for each $\chi \in A^o$, we have $|\chi(Y)| \leq \frac{1}{m}|\chi(A)| \leq \frac{1}{4m} < \varepsilon$. Thus $A^o \subset W$. ∎

By $N_o(G)$ we denote the family of all neighbourhoods of zero in an abelian topological group G (we do not assume neighbourhoods to be open).

(1.4) LEMMA. A character χ of an abelian topological group G is continuous if and only if $\chi \in U^o$ for a certain $U \in N_o(G)$.

Proof. The necessity of the condition is trivial. To prove the sufficiency, choose any $\varepsilon > 0$. We can find an integer $m > (4\varepsilon)^{-1}$ and then some $W \in N_o(G)$ with $W^m \subset U$. By (1.2), we have $|\chi(W)| \leqq \frac{1}{m}|\chi(U)| \leqq \frac{1}{4m} < \varepsilon$. This means that χ is continuous at zero. ▪

(1.5) PROPOSITION. The polars of neighbourhoods of zero in an abelian topological group G are compact subsets of $G^{\wedge}_{pc}$.

Proof. Choose any $U \in N_o(G)$. The group $G^{\wedge}_p$ is compact because we may identify it with a closed subgroup of the product T^G (see also (1.8)). Since U^o_p is a closed subset of $G^{\wedge}_p$, it is enough to show that the identity mapping $U^o_p \to U^o_{pc}$ is continuous.

Choose any $\kappa \in U^o$ and let W be a neighbourhood of κ in U^o_{pc}. By (1.3), there is some precompact subset A of G such that

$$W' := (\kappa + A^o) \cap U^o \subset W.$$

Next, we can find some $V \in N_o(G)$ with $V^3 \subset U$. Since A is precompact, there exist some $g_1,\dots,g_n \in A$ such that

(1) $\quad A \subset \{g_i\}_{i=1}^n + V.$

The set

$$W'' = \{\chi \in U^o : |\chi(g_i) - \kappa(g_i)| \leqq \frac{1}{12} \quad \text{for} \quad i = 1,\dots,n\}$$

is a neighbourhood of κ in U^o_p. It remains to show that $W'' \subset W'$.

So, choose any $\chi \in W''$. We have to show that $\chi - \kappa \in A^o$. Take any $g \in A$. In view of (1), we may write $g = g_i + h$ for some $i = 1,\dots,n$ and some $h \in V$. Now, from (1.2) we obtain

$$|\chi(V)| \leqq \frac{1}{3}|\chi(U)| \leqq \frac{1}{12} \quad \text{and} \quad |\kappa(V)| \leqq \frac{1}{3}|\kappa(U)| \leqq \frac{1}{12}.$$

Hence

$$|(\chi - \kappa)(g)| \leqq |\chi(g_i) - \kappa(g_i)| + |\chi(h)| + |\kappa(h)| \leqq \frac{3}{12} = \frac{1}{4}. \quad ▪$$

An abelian group G is called <u>divisible</u> if to each $g \in G$ and each $n = 1,2,\dots$ there corresponds some $h \in G$, with $nh = g$.

(1.6) PROPOSITION. Let H be a subgroup of an abelian group G. Every homomorphism of H into a divisible group can be extended to a homomorphism of G.

For the proof, see e.g. [38], Theorem (A.7).

Let G,H be abelian topological groups. An isomorphism ϕ of G onto H is called a topological isomorphism if ϕ and ϕ^{-1} are continuous. If there is a topological isomorphism of G onto H, then we say that G and H are topologically isomorphic and write $G \cong H$. An injective homomorphism $\phi : G \to H$ is called a topological embedding if ϕ is a topological isomorphism of G onto the group $\phi(G)$ endowed with the topology induced from H.

(1.7) PROPOSITION. Let $G_1,\ldots,G_n$ be abelian topological groups. There is a canonical topological isomorphism between $(G_1 \times \ldots \times G_n)^\frown_c$ and $(G_1)^\frown_c \times \ldots \times (G_n)^\frown_c$.

This is a standard fact. For the proof, we refer the reader to [38], (23.18) - the assumption there that $G_1,\ldots,G_n$ are locally compact is inessential. For infinite products, see (14.11) below.

Let H be a subgroup of an abelian topological group G. We say that H is <u>dually closed</u> in G if to each $g \in G \setminus H$ there corresponds some $\chi \in H^o$ with $\chi(g) \neq 0$ (this is equivalent to the assertion that H is a quasi-convex subset of G). Next, we say that H <u>dually embedded</u> in G if each continuous character of H can be extended to a continuous character of G. Observe that dually closed subgroups are closed. Observe also that each continuous character of H can be extended in a unique way to a continuous character of $\overline{H}$.

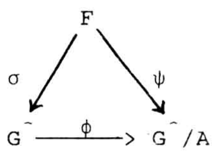
Let us recall shortly basic facts concerning the Pontryagin - van Kampen duality theorem. The proofs can be found e.g. in [38], §24. By a compact (resp. locally compact) group we shall mean a group which is compact (resp. locally compact) and separated. Locally compact abelian groups are called LCA groups.

(1.8) PROPOSITION. Let G be an LCA group. Then $G^\frown_c = G^\frown_{pc}$ is an LCA group, too, and the evaluation map is a topological isomorphism of G onto $(G^\frown_c)^\frown_c$. If H is a closed subgroup of G, then H is dually closed and dually embedded. Moreover, the canonical mappings $G^\frown_c/H^o_c \to H^\frown_c$ and $(G/H)^\frown_c \to H^o_c$ are both topological isomorphism. If G

is compact, $\hat{G}_c$ is discrete. If G is discrete, $\hat{G}_c$ is compact. There are canonical topological isomorphisms $\hat{R}_c \to R$, $\hat{Z}_c \to T$, $\hat{T}_c \to Z$.

(1.9) PROPOSITION. Let G be an LCA group. Then there exist an $n = 0,1,2,\ldots$, a compact group K and a discrete group D, such that G is topologically isomorphic to a closed subgroup of $R^n \times K \times D$.

Proof. Being an LCA group, $\hat{G}_c$ contains an open subgroup $A \cong R^n \times H$ for some $n = 0,1,2,\ldots$ and some compact group H ([69], Theorem 25 or [38], (9.14)). Let $\phi : \hat{G} \to \hat{G}/A$ be the natural projection. Every (abelian) group is a quotient of a free one. So, we can find a free abelian group F and a homomorphism ψ of F onto $\hat{G}/A$. Let $\{f_i\}_{i\in I}$ be a system of free generators of F. For each $i \in I$, choose some $\chi_i \in \hat{G}$ with $\phi(\chi_i) = \psi(f_i)$. Let $\sigma : F \to \hat{G}$ be the homomorphism given by $\sigma(f_i) = \chi_i$ for $i \in I$. We obtain the following commutative diagram:



The formula

$$\rho(a,f) = a + \sigma(f) \qquad (a \in A, \quad f \in F)$$

defines a homomorphism $\rho : A \times F \to \hat{G}$. We shall prove that

(1) $\quad \rho(A \times F) = \hat{G}.$

To this end, choose an arbitrary $\chi \in \hat{G}$. Since $\psi(F) = \hat{G}/A$, we can find some $f \in F$ with $\psi(f) = \phi(\chi)$. Then $\phi(\sigma(f)) = \psi(f) = \phi(\chi)$, which means that $a := \chi - \sigma(f) \in \ker \phi = A$. Thus $\chi = a + \sigma(f) = \rho(a,f) \in \rho(A \times F)$, which proves (1).

Let us endow F with the discrete topology. Since $A \times \{0\}$ is an open subgroup of $A \times F$ and ρ is a topological isomorphism (in fact, an identity) of $A \times \{0\}$ onto the open subgroup A of $\hat{G}_c$, it follows that $\rho : A \times F \to \hat{G}_c$ is both continuous and open. Consequently, $\hat{G}_c \cong (A \times F)/\ker \rho$. So, in virtue of (1.8), we have

$$G \cong (\hat{G}_c)^\frown_c \cong ((A \times F)/\ker \rho)^\frown_c \cong (\ker \rho)^o_c.$$

In other words, G is topologically isomorphic to a closed subgroup of

$(A \times F)_c^{\frown}$. From (1.7) we get $(A \times F)_c^{\frown} \simeq A_c^{\frown} \times F_c^{\frown} \simeq R^n \times H_c^{\frown} \times F_c^{\frown}$ and it remains to observe that $F_c^{\frown}$ is compact and $H_c^{\frown}$ discrete. ▪

The completion of an abelian topological group G will be denoted by $\tilde{G}$. We shall identify G with a dense subgroup of $\tilde{G}$. The closures in $\tilde{G}$ of elements of any given base at zero in G form a base at zero in $\tilde{G}$ ([23], Ch. III, §3, Proposition 7).

(1.10) PROPOSITION. Let G_o be a dense subgroup of an abelian topological group G. Let H be the closure in G of a closed subgroup H_o of G_o and let $\phi : G \to G/H$ be the canonical projection. Then the canonical bijection $G_o/H_o \to \phi(G_o)$ is a topological isomorphism of G_o/H_o onto a dense subgroup of G/H.

This is Proposition 21 of [23], Ch. III, §2.

(1.11) PROPOSITION. Let G be an abelian topological group. If G is a k-space, then $G^{\frown}$ is a complete group.

Proof. The space T^G of continuous mappings from G to T is complete in the compact-open topology ([52], Ch. 7, Theorem 12). It remains to observe that $G^{\frown}$ is a closed subset of T^G. ▪

(1.12) PROPOSITION. Let G be an abelian group and $\mathcal{B}$ a family of subsets of G satisfying the following conditions:

(a) every member of $\mathcal{B}$ contains zero;

(b) to each $U \in \mathcal{B}$ there corresponds some $V \in \mathcal{B}$ with $-V \subset U$;

(c) to each $U \in \mathcal{B}$ there corresponds some $V \in \mathcal{B}$ with $V + V \subset U$.

Then there exists a unique topology τ on G compatible with the group structure, such that $\mathcal{B}$ is a base at zero for τ.

For the proof, see [23], Ch. III, §1.2.

Let $\{G_i\}_{i \in I}$ be a family of abelian topological groups indexed by a set I. The product of these groups is defined in the usual way; we denote it by $\prod_{i \in I} G_i$. It is evident that the product of a family of locally quasi-convex groups is locally quasi-convex.

Let $\{p_{ij} : G_i \to G_j;\ i,j \in I,\ i \geqq j\}$ be an inverse system of topological groups, that is to say, I is a directed set and, for each pair $i,j \in I$ with $i \geqq j$, a continuous homomorphism $p_{ij} : G_i \to G_j$ is given, such that $p_{ij} \cdot p_{jk} = p_{ik}$ if $i \geqq j \geqq k$. We define the limit of this system in the usual way, identifying it with the appropriate subgroup of the product $\prod_{i\in I} G_i$. If I is the set of positive integers, then we speak of an inverse sequence. Naturally, the limit of an inverse system of locally quasi-convex groups is locally quasi-convex. The product $\prod_{i\in I} G_i$ is canonically topologically isomorphic to the limit of the inverse system $\prod_{i\in K} G_i$ where K runs through all finite subsets of I and the projections p_{KL} for $K \supset L$ are defined in the usual way.

The limit G of the inverse system $\{p_{ij}: G_i \to G_j\}$ may be equal to zero. If, however, I is at most countable and all p_{ij} are onto, then also all projections $p_i : G \to G_i$ are onto. Kaplan [50], Lemma 4.6, proved that if $\{p_{ij} : G_i \to G_j\}$ is an inverse sequence of LCA groups such that $p_{ij}(G_i)$ is dense in G_j for all pairs i,j with $i \geqq j$, then also $p_i(G)$ is dense in G_i for every i.

Again, let $\{G_i\}_{i\in I}$ be a family of abelian topological groups. Their <u>direct sum</u>, denoted by $\sum_{i\in I} G_i$, is algebraically the subgroup of the product $\prod_{i\in I} G_i$, consisting of finite sequences (that is, an element $(g_i)_{i\in I}$ of $\prod_{i\in I} G_i$ belongs to $\sum_{i\in I} G_i$ if and only if $g_i = 0$ for all but a finite number of indices i). We shall consider on $\sum_{i\in I} G_i$ the asterisk and the rectangular topologies. To describe them, we have to introduce some additional notions.

Let U be a subset of an abelian group G. For each $g \in U$, we define

$$n_U = \sup \{n : kg \in U \quad \text{for} \quad k = 1,\dots,n\}$$

and $g/U = (n_U)^{-1}$. This means, in particular, that $g/U = 0$ if and only if $kg \in U$ for every k.

Let us suppose that, for each $i \in I$, we are given some $U_i \in \mathcal{N}_o(G_i)$. We denote

$$\sum_{i\in I} U_i = \{ (g_i)_{i\in I} \in \sum_{i\in I} G_i : g_i \in U_i \quad \text{for all} \quad i \in I\},$$

$$\sum_{i\in I}{}^{*} U_i = \{(g_i)_{i\in I} \in \sum_{i\in I} U_i : \sum_{i\in I} (g_i/U_i) < 1\}.$$

Let $\mathcal{B}$ be the family of all sets of the form $\sum_{i\in I} U_i$ where $U_i \in \mathcal{N}_o(G_i)$ for every i. Similarly, let $\mathcal{B}^*$ be the family of all sets of the form $\sum^*_{i\in I} U_i$ where $U_i \in \mathcal{N}_o(G_i)$ for every i. It follows from (1.12) that there is a unique topology on $\sum_{i\in I} G_i$ compatible with the group structure, for which $\mathcal{B}$ is a base at zero; we call it the rectangular topology. Conditions (a) - (c) of (1.12) are satisfied trivially. Similarly, it follows from (1.12) that there is a unique topology on $\sum_{i\in I} G_i$ compatible with the group structure, for which $\mathcal{B}^*$ is a base at zero; we call it the asterisk topology. The only non-trivial thing here is to verify (1.12) (c) with $\mathcal{B}$ replaced by $\mathcal{B}^*$:

(1.13) LEMMA. Let $\{G_i\}_{i\in I}$ be a family of abelian topological groups and let $\mathcal{B}^*$ be defined as above. Then to each $U \in \mathcal{B}^*$ there corresponds some $V \in \mathcal{B}^*$ with $V + V \subset U$.

To prove (1.13), we need the following simple proposition whose verification is left to the reader:

(1.14) LEMMA. Let U be a zero-containing subset of an abelian group G. Then $g/(U + U) \leq \frac{1}{2}(g/U)$ for each $g \in U$. If V is another zero-containing subset of G and $V + V \subset U$, then $(g + h) / U \leq \max (g/V, h/V)$ for all $g,h \in V$.

Proof of (1.13). Choose an arbitrary $U \in \mathcal{B}^*$. We have $U = \sum^*_{i\in I} U_i$ for some $U_i \in \mathcal{N}_o(G_i)$, $i \in I$. For each $i \in I$, we can find some $V_i \in \mathcal{N}_o(G_i)$ with $V_i^4 \subset U_i$. Set $V = \sum^*_{i\in I} V^i$.

Now, take any sequences $(g_i)_{i\in I}$ and $(h_i)_{i\in I}$ belonging to V. From (1.14) we get

$$\sum_{i\in I} [(g_i + h_i)/U_i] \leq \frac{1}{2} \sum_{i\in I} [(g_i + h_i)/(V_i + V_i)]$$

$$\leq \frac{1}{2} \sum_{i\in I} \max (g_i/V_i, h_i/V_i) \leq \frac{1}{2} \sum_{i\in I} [g_i/V_i + h_i/V_i] < 1.$$

Thus $(g_i + h_i)_{i\in I} \in \sum^*_{i\in I} U_i$, which means that $V + V \subset U$. ■

The asterisk topology is, by definition, finer than the rectangular one. For countable direct sums, these two topologies are identical:

(1.15) PROPOSITION. Let $(G_n)_{n=1}^{\infty}$ be a sequence of abelian topological groups. Then the asterisk topology on $\sum_{n=1}^{\infty} {}' G_n$ is equal to the rectangular one.

Proof. Let $U_n \in N_o(G_n)$ for $n = 1,2,\ldots$. We have to show that $\sum_{n=1}^{\infty}{}^* U_n$ contains a rectangular neighbourhood of zero. For each $n = 1,2,\ldots,$ we can find some $V_n \in N_o(G_n)$ with $V_n^2 \subset U_n$. From (1.14) it follows by induction that

$$(g/U_n) \leq 2^{-n}(g/V_n) \quad \text{for all} \quad n = 1,2,\ldots \quad \text{and all} \quad g \in V_n.$$

So, if $(g_n)_{n=1}^{\infty} \in \sum_{n=1}^{\infty} V_n,$ then

$$\sum_{n=1}^{\infty} (g_n/U_n) \leq \sum_{n=1}^{\infty} 2^{-n}(g_n/V_n) < \sum_{n=1}^{\infty} 2^{-n} = 1.$$

Thus $\sum_{n=1}^{\infty} V_n \subset \sum_{n=1}^{\infty}{}^* U_n$. ∎

In general, the rectangular topology is not equivalent to the asterisk one (consider, for instance, an uncountable direct sum of real lines). In the sequel, speaking of direct sums of topological groups, we shall always assume that they are endowed with the asterisk topology, unless it is explicitly stated otherwise. Notice that if $\{G_i\}_{i\in I}$ is a family of locally convex spaces, then the group $\sum_{i\in I} G_i$ is topologically isomorphic to their locally convex direct sum.

(1.16) PROPOSITION. The direct sum of an arbitrary family of locally quasi-convex groups is locally quasi-convex.

An easy proof is left to the reader.

(1.17) PROPOSITION. Let G be the direct sum of a family $\{G_i\}_{i\in I}$ of Hausdorff abelian groups. For each $i \in I$, let $\pi_i : G \to G_i$ be the canonical projection. If P is a precompact subset of G relative to the asterisk or rectangular topology, then $\pi_i(P) = \{0\}$ for all but finitely many indices i.

Proof. Suppose that P is precompact in the rectangular topology. Set $J = \{i \in I : \pi_i(P) \neq \{0\}\}$. We have to show that J is finite. Suppose the contrary. To each $i \in J$ there corresponds some $g_i \in P$ with $\pi_i(g_i) \neq 0$. Next, there is some $U_i \in N_o(G_i)$ with $\pi_i(g_i) \notin U_i$ because G_i is separated. The set $U = \sum_{i\in I} U_i$ is a rectangular neighbourhood of zero in G. So, there is a finite subset A of P such that $P \subset A + U$ because P is precompact. Since A is finite and consists of finite sequences while J is infinite, it follows that there is an index $j \in J$ such that $\pi_j(A) = \{0\}$. Then

$$\pi_j(P) \subset \pi_j(A + U) = \pi_j(A) + \pi_j(U) = \{0\} + U_j = U_j.$$

On the other hand, we have $g_j \in P$ and $\pi_j(g_j) \notin U_j$, which is a contradiction. ■

Let $\{p_{ij} : G_i \to G_j;\ i,j \in I,\ i \leqq j\}$ be a direct system of abelian topological groups, that is to say, I as a directed set and, for each pair $i,j \in I$ with $i \leqq j$, a continuous homomorphism $p_{ij} : G_i \to G_j$ is defined, such that $p_{ij} \cdot p_{jk} = p_{ik}$ if $i \leqq j \leqq k$. Let G be the direct sum of the family $\{G_i\}_{i \in I}$ and let G_o be the subgroup of G generated by all elements of the form

$$g_i - p_{ij}(g_i) \qquad (i,j \in I;\ i \leqq j;\ g_i \in G_i)$$

(we treat G_i and G_j as subgroups of G). We define the limit of the system considered as the quotient group G/G_o. When all groups G_i are locally convex spaces, we obtain the usual definition of the inductive limit.

Kaplan [50] defined the limit of the direct system as the group $G/\overline{G_o}$. He proved that if I is countable, the groups G_i are locally compact and all mappings p_{ij} are injective, then G_o is closed ([50], Theorem 8, p. 433).

It is not hard to see that if J is a cofinal subset of I and all groups G_i are locally quasi-convex, then the limit of the system $\{p_{ij} : G_i \to G_j;\ i,j \in I,\ i \leqq j\}$ may be identified with the limit of the subsystem $\{p_{ij} : G_i \to G_j;\ i,j \in J,\ i \leqq j\}$. The assumption of local quasi-convexity is essential.

If I is the set of positive integers, then we speak of direct sequences. In view of the last remark, when considering limits of countable direct systems we may restrict ourselves to limits of direct sequences.

The direct sum of a family $\{G_i\}_{i \in I}$ of locally quasi-convex groups is easily seen to be topologically isomorphic to the limit of the direct system $\{p_{KL} : \sum_{i \in K} G_i \to \sum_{i \in L} G_i\}$ where K,L run through finite subsets of I and the embeddings p_{KL} are defined in the usual way.

(1.18) PROPOSITION. Let G be the limit of a direct sequence $\{p_n : G_n \to G_{n+1}\}$ of abelian topological groups, in which all mappings p_n are topological embeddings. Then the topology of G induces original topologies on the groups G_n. Consequently, if all groups G_n are separated, so is G.

Proof. We may assume that $(G_n)_{n=1}^{\infty}$ is an increasing sequence of subgroups of G. Let $\mathcal{B}$ be the family of all sets of the from

$$U_1 + U_2 + \ldots := \bigcup_{n=1}^{\infty} (U_1 + \ldots + U_n)$$

where $U_n \in \mathcal{N}_o(G_n)$ for every n. It follows directly from (1.15) that $\mathcal{B}$ is a base at zero in G.

Fix an index m and choose an arbitrary $V \in \mathcal{N}_o(G_m)$. We are to find some $U \in \mathcal{N}_o(G)$ with $U \cap G_m \subset V$. Naturally, we may assume that $m = 1$. There is some $U_1 \in \mathcal{N}_o(G_1)$ with $U_1 + U_1 \subset V$, and a simple inductive argument allows us to find, for each $n \geq 2$, some $U_n, W_n \in \mathcal{N}_o(G_n)$ with $W_n \cap G_{n-1} \subset U_{n-1}$ and $U_n + U_n \subset W_n$. It remains to show that $G_1 \cap (U_1 + U_2 + \ldots) \subset V$. Set $Y_k = U_1 + \ldots + U_k + U_k$ for $k = 1,2,\ldots$. It is enough to show that $G_1 \cap Y_k \subset V$ for every k. For $k = 1$, this is obvious. For $k > 1$, we use induction:

$$\begin{aligned} G_1 \cap Y_k &\subset G_1 \cap G_{k-1} \cap (U_1 + \ldots + U_{k-1} + W_k) \\ &= G_1 \cap (U_1 + \ldots + U_{k-1} + (W_k \cap G_{k-1})) \\ &\subset G_1 \cap (U_1 + \ldots + U_{k-1} + U_{k-1}) = G_1 \cap Y_{k-1}, \end{aligned}$$

which is contained in V due to the inductive assumption. ∎

A topological vector space is locally convex if and only if it is a Hausdorff locally quasi-convex group (see (2.4)). Komura [55] showed that the limit of an uncountable direct system $\{p_{ij} : E_i \to E_j\}$ of locally convex spaces in which all mappings p_{ij} are topological embeddings need not be locally convex, and even if it is, it need not induce original topologies on the spaces E_i. If, in (1.18), all groups G_n are locally convex spaces, G is a locally convex space, too ([80], Ch. II, (6.4)). If we assume only that all G_n's are Hausdorff locally quasi-convex groups, then probably G need not be locally quasi-convex. See, however, (7.9). Vilenkin [99] considered another topology on the limit of a direct system. Under his definition, the limit of <u>any</u> direct system of abelian topological groups is a locally quasi-convex group.

Let G, H be abelian topological groups and let $\phi : G \to H$ be a continuous homomorphism. Then the formula

$$\langle \psi(\chi), g \rangle = \langle \chi, \phi(g) \rangle \qquad (\chi \in H^\frown;\ \ g \in G)$$

defines a homomorphism $\psi : H^\frown \to G^\frown$. We call it the <u>dual homomorphism</u> and denote by $\phi^\frown$. It is clear that $\phi^\frown : H^\frown_\tau \to G^\frown_\tau$ is continuous when τ is the topology of pointwise (resp. compact, precompact) convergence. If $\{p_{ij} : G_i \to G_j;\ i,j \in I;\ i \geq j\}$ is an inverse system of abelian topological groups, then $\{p^\frown_{ij} : G^\frown_j \to G^\frown_i\}$ is a direct system, and vice versa.

Let $\{G_i\}_{i \in I}$ be a family of abelian topological groups indexed by a set I. Suppose that, for each $i \in I$, a closed subgroup H_i of G_i has been chosen. Let G be the subgroup of the product $\prod_{i \in I} G_i$ consisting of all sequences $(g_i)_{i \in I}$ such that $g_i \in H_i$ for all but finitely many indices i. We topologize G by identifying it with the limit of the inverse system

$$\pi_{KL} : \sum_{i \in K} G_i \times \sum_{i \notin K} (G_i/H_i) \to \sum_{i \in L} G_i \times \sum_{i \notin L} (G_i/H_i)$$

where K, L are finite subsets of I with $K \supset L$, and π_{KL} is the canonical projection. Endowed with this topology, G will be called the <u>reduced product</u> of groups G_i relative to subgroups H_i and denoted by $\sum_{i \in I} (G_i : H_i)$. It is clear that the topology of G induces original topologies on the subgroups G_i. Notice that if $H_i = G_i$ for almost all i, then $G = \prod_{i \in I} G_i$; if $H_i = \{0\}$ for almost all i, then $G = \sum_{i \in I} G_i$. More precisely, if H is the subgroup of G consisting of all $(g_i)_{i \in I}$ such that $g_i \in H_i$ for every i, then H has the usual product topology and G/H is topologically isomorphic to the direct sum $\sum_{i \in I} (G_i/H_i)$. Since inverse limits and direct sums of locally quasi-convex groups are locally quasi-convex, G is locally quasi-convex provided that so are all groups G_i and G_i/H_i.

(1.19) PROPOSITION. For each $i \in I$, let $\psi_i : G \to G_i$ be the canonical projection. A subset X of G is precompact if and only if $\psi_i(X)$ is precompact in G_i for all i, and $\psi_i(X) \subset H_i$ for almost all i.