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Point Set Topology Conference 1971

Houston, Texas

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UNIVERSITY OF HOUSTON

POINT SET TOPOLOGY CONFERENCE

MARCH 22, 23, 24, 1971

EDITED BY: D. REGINALD TRAYLOR

HOUSTON, TEXAS

PREFACE

The University of Houston hosted a conference on Point Set Topology, March 22 - 24, 1971. Meetings were held at the Shamrock Hilton Hotel. The program consisted of twenty-six speakers and was attended by over one hundred participants.

These Proceedings include papers which were presented at that conference. Results announced at the conference, but not included herein, are expected to appear elsewhere.

The speakers were:

Armentrout, S. (Pennsylvania State University)

Bandy, C. (University of Houston)

Bourgin, D. (University of Houston)

Eberhart, C. (University of Kentucky)

Fugate, J. (University of Kentucky)

Heath, R. (University of Pittsburgh)

Hodel, R. (Duke University)

Jones, F. B. (University of California at Riverside)

Kropa, J. (Emory University)

Kuperberg, W. (University of Stockholm, Sweden)

Lelek, A. (University of Houston)

Michael, E. (University of Washington)

Mohler, L. (University of Saskatchewan, Canada)

Nadler, S. (Loyola University)

Nagata, J. (University of Pittsburgh)

O'Steen, D. (University of Houston)

Riecke, C. (University of Houston)

Rogers, J. T., Jr. (Tulane University)

Rudin, M. E. (University of Wisconsin)

Stevenson, N. (State University of New York at Binghamton)

Stone, A. (University of Rochester)

Transue, W. (Auburn University)

Van Doren, K. (Auburn University)

Wicke, H. (Ohio University and Sandia Corporation)

Williams, S. (Pennsylvania State University)

Zenor, P. (Auburn University)

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SOME NEW ASPECTS IN CURVE THEORY

by

A. Lelek

The set-theoretical approach to the structure of continua brings us to some problems concerning curves. By a continuum we understand to mean any compact connected metric space, and a curve means any one-dimensional continuum. The following class of curves has been introduced in [3]. We call a curve Suslinian provided each collection of pairwise disjoint subcurves of it is countable. Thus each rational curve is Suslinian, and each Suslinian curve is hereditarily decomposable. We say that a curve X is acyclic provided each continuous mapping of X into the circle is homotopic to a constant mapping. A hereditarily decomposable curve X is acyclic if and only if X is hereditarily unicoherent. We call a space hereditarily discontinuous provided each continuum contained in it is degenerate. It is known [3] that an acyclic curve X is Suslinian if and only if X admits a decomposition $X = P \cup Q$ where P is hereditarily discontinuous and Q is countable. Clearly, the existence of such a decomposition is a sufficient condition for any curve X to be Suslinian. The problem which follows has been raised much earlier than it has been published in [4], and still remains unsettled.

Problem I. Does each Suslinian curve X admit a decomposition $X = P \cup Q$ where P is hereditarily discontinuous and Q is countable?

Suppose X is a Suslinian curve which does not admit the decomposition required in Problem I. Then we know [1] the following condition (c) is satisfied.

(c) There exists a point $p \in X$ and two infinite sequences of continua $C_n \subset X$ and $K_n \subset X$ such that

$$p \in C_n, \qquad C_{n+1} \subset C_n, \qquad \operatorname{diam} C_n < \frac{1}{n},$$

$$p \notin K_n, \qquad K_n \cap K_m = \emptyset, \qquad \operatorname{diam} K_n < \frac{1}{n},$$

and $C_n \cap K_n \neq \emptyset \neq K_n \setminus C_1$ for all positive integers n and m $(m \neq n)$.

As a consequence, we can say that similarly to the case when the curve X is acyclic the answer to Problem I is "yes" also in the case when X is atriodic, i.e. when for each three subcurves C_1, C_2, C_3 of X such that

$$C_o = C_1 \cap C_2 = C_2 \cap C_3 = C_1 \cap C_3 \neq \emptyset$$

is connected, C_o coincides with at least one of the curves C_1, C_2, C_3. This is because condition (c) does not hold for atriodic curves. However, we have a stronger result in the latter case. We call a space hereditarily disconnected provided each connected set contained in it is degenerate. Thus each hereditarily disconnected space is hereditarily discontinuous, but there exist hereditarily discontinuous spaces which are even connected and non-degenerate. It is known [1] that each atriodic Suslinian curve X admits a decomposition $X = P \cup Q$ where P is hereditarily disconnected and Q is countable. On the other hand, there exists [1] an acyclic Suslinian curve X such that $X \neq P \cup Q$ where P is not connected and Q is countable. There also exists [1] a chainable Suslinian curve X such that $X \neq P \cup Q$ where P is zero-dimensional and Q is countable, i.e. the curve X is not rational. An example of a Suslinian dendroid which is not rational has been constructed earlier in [3]. The reader is advised to consult [4] for a discussion concerning decomposition properties of curves.

Problem II. Does each Suslinian curve X for which condition (c) does not hold admit a decomposition $X = P \cup Q$ where P is hereditarily disconnected and Q is countable?

The rim-type of a rational curve X is understood to mean the minimum ordinal α such that X possesses an open basis consisting of sets with countable boundaries whose α-th derivatives are empty. Some historical remarks about this notion are to be found in [5]. For instance, there exist examples of chainable rational curves of rim-type ω which contain no arcs. It is known [5] that each acyclic rational curve of finite rim-type contains an arc. Of course, the arc is the simplest example of a curve of rim-type 1. For any integer $n \geq 2$, there exists [2] a chainable rational curve X_n such that each subcurve of X_n has rim-type 1 or n.

Problem III. Does each rational curve of finite rim-type contain an arc?

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[1] H. Cook and A. Lelek, On the topology of curves IV, Fund. Math. (to appear)

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[3] A. Lelek, On the topology of curves II, Fund. Math. 70(1971), 131-138.

[4] ________, Some problems concerning curves, Colloq. Math. 23(1971), 93-98.

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UNIVERSITY OF HOUSTON

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CUTS AND WEAK CUTS IN METRIC CONTINUA

by

L. Mohler (Buffalo)

A closed subset C of a topological space X is said to cut X between two of its points p and q if X-C does not contain a connected subset containing both p and q. (note that C cuts X between some pair of its points if and only if X-C is disconnected). C is said to weakly cut X between p and q if X-C does not contain a continuum containing both p and q (note that C weakly cuts X between some pair of its points if and only if X-C is not continuumwise connected). C is said to be a cutting of X if it cuts X between some pair of its points and is said to be a weak cutting of X if it weakly cuts X between some pair of its points.[1]

Clearly every cutting of a space is a weak cutting, but in general, the two notions are different. However, if X is a locally connected continuum (a continuum is a compact connected metric space), then every weak cutting of X is a cutting of X. This follows from the fact that every open connected subset of X is arcwise connected. Thus if a closed subset of X fails to cut X, it also fails, to weakly cut X.

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[1] Nomenclature for these notions is not standardized. In particular the word "cutting" is sometimes used to mean "weak cutting".

In [3][2] Knaster raised the question of whether or not the equivalence of these two notions characterizes the locally connected continua among all continua. That it, suppose that X is a continuum and every weak cutting of X is also a cutting of X. Does it follow that X is locally connected?[3] In [6] the author showed that this is in fact the case. Indeed the following (slightly more general) theorem holds:

Theorem 1 : If X is a generalized continuum,[4] then X is locally connected if and only if every weak cutting of X is a cutting of X.

The theorem is proved by contradiction. Assuming that X is not locally connected (but that every weak cutting is a cutting) one produces a family of candidates for weak cuttings of X which are not cuttings by removing certain boundary points from the closures of components of neighborhoods of a point of non-local connectedness. A theorem of Whyburn on non-separated families of cuttings ([8], p. 45, th. 2.2) is then applied to show that at least one of these sets is a weak cutting but not a cutting.

An equivalent statement of theorem 1 can be obtained by looking at the complements of non-cuttings of X :

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[2] See also [2].

[3] The question was probably suggested by the following theorem which appeared in Zarankiewicz' doctoral dissertation [10] : A continuum X is locally connected if and only if for every pair of points p and q in X and for every closed subset C of X, C cuts X between p and q if and only if C weakly cuts X between p and q.

[4] A generalized continuum is a locally compact, connected separable metric space. It has been pointed out to the author by J.H.V. Hunt that this definition is redundant; since any locally compact, connected metric space is necessarily separable. See [1].

Theorem 1': If X is a generalized continuum, then X is locally connected if and only if every connected open subset of X is continuumwise connected.

Using this statement of the theorem, the author [7] has obtained a number of characterizations of hereditarily locally connected continua (continua each of whose subcontinua is locally connected). For example :

Theorem 2 : A continuum X is hereditarily locally connected if and only if every connected, locally compact subset of X is arcwise connected.

One might guess that if X is a hereditarily locally connected continuum, then every connected subset of X is arcwise connected (Wilder [9] has shown that every connected subset of X is locally connected). However, this is not the case. In [4] Knaster and Kuratowski give an example of a regular (and hence hereditarily locally connected) continuum which contains a connected subset with no compact perfect subsets (and hence with no non-trivial subcontinua). The following problem thus suggests itself :

Problem: Characterize those continua in which every connected subset is arcwise connected. Are all such continua regular?

Kuratowski ([5], p. 273, th. 10) has shown that every connected subset of a continuum X is arcwise connected if and only if every connected subset of X is continuumwise connected.

It is perhaps interesting to note that theorem 1 fails in the non-metric setting. In [6] an example is given of a compact connected Hausdorff space each of whose connected open subsets is arcwise connected, but which fails to be locally connected.

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