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William Arveson

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by

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TEN LECTURES ON OPERATOR ALGEBRAS

Introduction

This book contains somewhat expanded versions of ten lectures delivered at Texas Tech University during the summer of 1983. The operator algebras of the title are nonselfadjoint algebras of operators on Hilbert space.

This subject is new, and has shown remarkable growth in the last twenty years. Indeed, when I was finishing my graduate studies in 1964 I knew of only three papers that addressed themselves seriously to nonselfadjoint operator algebras ([**42**, **61**], and a paper of John Schue, *The structure of hyperreducible triangular algebras*, Proc. Amer. Math. Soc. **15** (1964), 766–772). A few of us believed in the sixties that this was a promising way to approach the theory of single operators, but we certainly did not see how and were not even sure if that would be accomplished. What actually happened was that the subject developed in several directions, and was pursued entirely on its own merits. When the applications to single operator theory did come, they came unexpectedly and in surprising ways (see Lecture 10). These results are deep and, looking back on it now, I must say that I cannot conceive of any way that the methodology of single operator theory could have produced them.

The subject matter for these lectures has been selected using subjective criteria. Some of it has historical interest, some of it seems timely or important (at least to me), some of it seems to suggest new directions, and some of it is just fun to communicate. I have had to omit several of my favorite topics on which there has been significant progress, including noncommutative Silov boundaries, abstract dilation theory, and algebras defined by group actions [**7, 8, 73, 74, 50, 12, 49**].

Some of the material is expository and is presented largely without proofs (Lectures 1, 2, 5, 6). Some is expository but with complete proofs or complete ideas of proofs (Lectures 7, 8, 10). Lectures 7 and 8 expand on some notes distributed to the participants in a seminar at Berkeley during the spring quarter 1983; Lecture 9 is based on a lecture delivered in Busteni, Romania, in September 1983. Lecture 4 contains new material relating to the Feynman-Kac formula. I have taken some care to present complete proofs, and to develop the background material from classical mechanics and quantum mechanics in Lecture 3.

Finally, the references are by no means complete. I have referenced only those items I know about that relate to the subject matter of these lectures. The reader will find a more complete bibliography in the survey of John Erdos [**29**], which also contains a discussion of several topics not mentioned in these lectures.

I would like to thank the National Science Foundation for granting financial support to this lecture series, the conference organizer, Gareth Ashton, for his efforts to make the conference work, and the Mathematics Department of Texas Tech University for the hospitality they extended to all the participants.

Note added December 5, 1983, *concerning Lectures* 4 *and* 5. Barry Simon has pointed out that connections between the Feynman-Kac formula and dilation theory have been observed previously in the mathematical physics literature. We particularly want to call attention to the work of Abel Klein [**75, 76, 77, 78**] on Osterwalder-Schrader positivity, and of Klein and Landau [**79, 80**] on the path space approach to perturbation theory and KMS systems.

Contents

Lecture 1. Origins in Single Operator Theory

Many of the concepts that are now a basic part of this subject (triangular operator algebras, compact perturbations, quasitriangular operator algebras) have identifiable origins in single operator theory. Others do not (e.g., the contents of Lectures 4–8). In this lecture I want to describe some ideas from single operator theory that have led to significant generalizations in the theory of operator algebras. I will also discuss some ways of thinking about these things that I have found to be useful.

Consider first the case of an operator T on a finite-dimensional Hilbert space $\mathcal{H}$. Every graduate student knows that there is an orthonormal basis $\{e_1, e_2, \dots, e_n\}$ for $\mathcal{H}$ relative to which the matrix of T is upper triangular. Equivalently, there exist (selfadjoint) projections $0 = P_0 < P_1 < \cdots < P_n = 1$ such that

(1.1)
(i) $(1 - P_k)TP_k = 0, 0 \leqslant k \leqslant n,$
(ii) *the algebra generated by* $\{P_k\}$ *is maximal abelian.*

The key step in the proof consists of showing that every operator on a nonzero finite-dimensional Hilbert space has an eigenvector—a one-dimensional invariant subspace. The triangular form (1.1) is obtained by repeatedly applying that result to the projections of T onto the various quotients

$$\mathcal{H}/P_k\mathcal{H}, \qquad k = 0, 1, \dots, n-1,$$

as the P_k's are constructed one by one. Nothing like this is known for operators on infinite-dimensional Hilbert spaces; indeed it is not even known if general operators have even a single nontrivial invariant subspace.

In order to discuss this further, let us call an operator T (or a set of operators $\{T_\alpha\colon \alpha \in I\}$) *transitive* if the only closed subspaces $\mathcal{M}$ of the underlying Hilbert space $\mathcal{H}$ satisfying

$$T\mathcal{M} \subseteq \mathcal{M} \qquad (\textit{resp. } T_\alpha\mathcal{M} \subseteq \mathcal{M} \textit{ for all } \alpha)$$

are $\mathcal{M} = \{0\}$ or $\mathcal{M} = \mathcal{H}$; the term *intransitive* simply means not transitive. The following result is an unpublished theorem attributed to von Neumann. A generalization to Banach spaces was found by Aronszajn and Smith [**4**].

THEOREM 1.2. *Every compact operator on infinite-dimensional Hilbert space is intransitive.*

Responding to a conjecture of Kennan Smith which was popularized by Paul Halmos, A. Bernstein and A. Robinson [**17**] found a significant generalization of Theorem 1.2 in which the hypothesis that T is compact is replaced by the hypothesis that $p(T)$ is compact for some nontrivial polynomial p. Their proof had metamathematical aspects which made many functional analysts uncomfortable, and soon Halmos published a somewhat improved "translation" of their proof into more conventional operator theoretic terms [**36**]. The latter was generalized and simplified in [**16**], and in short order a flurry of papers had inundated the subject. Years later, Lomonosov found a dramatic generalization: *if an operator algebra $\mathcal{A}$ commutes with a nonzero compact operator, then $\mathcal{A}$ is intransitive.* Lomonosov's method was entirely new, and seemingly, his result had rendered obsolete much of the preceding work on invariant subspaces.

While that statement is true in some limited sense, it is certainly misleading. And there is a lesson here. What has survived from the pre-Lomonosov methodology is a concept (quasitriangularity). This concept has suggested new problems, and new formulations of old problems, which have led to remarkable progress in single operator theory and in operator algebras.

Let us begin by sketching the essential idea behind the proof in [**16**]. One is given a quasinilpotent operator T such that $p(T) \neq 0$ for every polynomial $p \neq 0$, and a cyclic vector ξ for T. Let P_n be the n-dimensional projection onto $[\xi, T\xi, T^2\xi, \ldots, T^{n-1}\xi]$. P_n is not invariant under T, but a routine computation shows that the sequence $\{P_n \colon n \geqslant 1\}$ is asymptotically invariant in the following sense:

$$\lim_{n \to \infty} \|(1 - P_n)TP_n\| = 0, \tag{1.2}$$

[**16**, p. 61]. If the norm-closed algebra generated by 1 and T contains a nonzero compact operator, then one can construct a nontrivial invariant subspace for T. Briefly, one finds a judiciously chosen sequence Q_n of invariant projections for the sequence of finite-dimensional operators

$$T_n = P_n T|_{\operatorname{ran} P_n}.$$

One then extracts a subsequence $Q_{n_1}, Q_{n_2}, \ldots$ which converges weakly to a positive selfadjoint operator Q. If one makes the "right" choice of $\{Q_n\}$, then it can be shown that $\mathcal{M} = \{\xi \colon Q\xi = \xi\}$ is a nontrivial T-invariant subspace.

Halmos made the condition (1.2) into a definition, thereby initiating the theory of quasitriangular operators [**37**]. Let us recall the basic definitions. An operator $T \in \mathcal{L}(\mathcal{H})$ is called *triangular* if there is an increasing sequence $P_1 \leqslant P_2 \leqslant \cdots$

of finite-dimensional projections such that $P_n \to 1$ strongly, and

$$(1 - P_n)TP_n = 0, \qquad n = 1, 2, \ldots.$$

The dimensions of the projections P_n are allowed to increase arbitrarily fast. However, since the restriction of T to the range of each P_n is a finite-dimensional operator, one may use conventional linear algebra to find a new sequence $\{P_n'\}$ of invariant projections, which refines the original sequence $\{P_n\}$ in the sense that $\{P_n\} \subseteq \{P_n'\}$, which satisfies $P_n' \leqslant P_{n+1}'$, and is such that P_n' is n-dimensional for every n.

CONCLUSION. *An operator T is triangular iff there is an orthonormal basis $\{e_1, e_2, \ldots\}$ with respect to which the matrix of T is upper triangular*:

$$T \sim \begin{bmatrix} * & * & * & \cdot & \cdot & \cdot \\ 0 & * & * & \cdot & \cdot & \cdot \\ 0 & 0 & * & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & & & \\ \cdot & \cdot & \cdot & & & \\ \cdot & \cdot & \cdot & & & \end{bmatrix}.$$

Let us write $\mathcal{T}$ for the class of all triangular operators (I am going to ignore the obvious set-theoretic difficulties associated with such a definition, leaving it for the reader to reformulate the definition of $\mathcal{T}$ so as to obtain a *bona fide* set).

An operator $T \in \mathcal{L}(\mathcal{H})$ is called *quasitriangular* if there is an increasing sequence $\{P_n\}$ of finite-dimensional projections in $\mathcal{L}(\mathcal{H})$ such that $P_n \to 1$ strongly and

$$\|(1 - P_n)TP_n\| \to 0,$$

as $n \to \infty$. Let us write $\mathcal{QT}$ for the class of all quasitriangular operators.

Now if $\{P_n\}$ is any sequence of projections which tends strongly to the identity, then one can show easily that

$$\lim_{n \to \infty} \|(1 - P_n)KP_n\| = 0$$

for every compact operator K. It follows that every compact perturbation of a triangular operator is quasitriangular. Significantly, these two classes of operators actually coincide [**37**].

THEOREM 1.3. $\mathcal{QT} = \mathcal{T} + \mathcal{K}$.

SKETCH OF PROOF. Let A be a quasitriangular operator and let $\{P_n\}$ be a sequence of finite rank projections which increases to 1 and satisfies

$$\lim_{n \to \infty} \|(1 - P_n)AP_n\| = 0. \tag{1.4}$$

We may find a subsequence $P_{n_1}, P_{n_2}, \ldots$ of $\{P_n\}$ so that the numbers $\|(1 - P_{n_k})AP_{n_k}\|$ tend to zero as fast as we like, and in particular there is a subsequence $Q_1 = P_{n_1}, Q_2 = P_{n_2}, \ldots$ such that

$$\sum_{k=1}^{\infty} \|(1 - Q_k)AQ_k\| < \infty. \tag{1.5}$$

(1.5) implies that the operator

$$K = \sum_{k=1}^{\infty} (1 - Q_k) A (Q_k - Q_{k-1})$$

is an absolutely convergent series of finite rank operators, and is therefore compact. Moreover, the operator $T = A - K$ is triangular because K has been constructed so that

$$(1 - Q_k) K Q_k = (1 - Q_k) A Q_k,$$

and hence $(1 - Q_k) T Q_k = 0$, $k = 1, 2, \ldots$. Therefore $A = T + K$ belongs to $\mathcal{T} + \mathcal{K}$. □

The above argument shows that if $\{P_n\}$ is a sequence of finite rank projections which increases to 1 and A is an operator such that $\|(1 - P_n) A P_n\| \to 0$, then A has a decomposition

$$A = T + K \tag{1.6}$$

as a compact perturbation of an operator T which leaves an *infinite subsequence* $\{P_{n_k}\}$ *of* $\{P_n\}$ *invariant*. A natural question here is whether or not T and K can be chosen so that T leaves the *entire* sequence $\{P_n\}$ invariant. The answer turns out to be yes but the proof is quite different from the proof of Theorem 1.3, and involves operator algebraic techniques which will be discussed in Lecture 2 (Theorem 2.10).

We will return to quasitriangular operators presently. I want to digress for a time in order to describe some ways of thinking about compact perturbations that have been useful to me. Consider first the space l^∞ of all bounded complex-valued sequences $a = \{a_n: n = 1, 2, \ldots\}$. l^∞ is a commutative C^*-algebra with unit relative to the usual norm and operations: for instance, the product of two sequences a and b is the sequence $\{a_n b_n: n = 1, 2, \ldots\}$.

What does it mean for two sequences to have the "same properties"? We mean by this that there should be an automorphism of the given C^*-algebra structure of l^∞ which carries one sequence to the other. It is not hard to determine the group of all automorphisms of l^∞. The most general automorphism α is given by a permutation π of the set N of positive integers as follows:

$$\alpha(a) = \{a_{\pi(n)}: n \in N\}, \qquad a \in l^\infty. \tag{1.7}$$

The reason is that an automorphism α of l^∞ must permute the minimal projections of l^∞, the latter are identified as the characteristic functions of singleton subsets of $\mathbf{N}$, and thus we obtain a permutation π of $\mathbf{N}$ satisfying (1.7).

What does it mean for two sequences a and b to have the "same asymptotic properties"? It is reasonable to require this to mean that there should be an automorphism α of l^∞ such that $\alpha(a) - b$ *vanishes at infinity*. Thus one may introduce an equivalence relation as follows:

DEFINITION 1.8. *$a \sim b$ iff there is a permutation π of $\mathbf{N}$ such that*

$$\lim_{n \to \infty} |a_{\pi(n)} - b_n| = 0.$$

Consider the subspace c_0 of l^∞ consisting of all sequences a satisfying $\lim_{n\to\infty} a_n = 0$. c_0 is a closed selfadjoint ideal in l^∞, and so we may form the quotient C^*-algebra l^∞/c_0 and the canonical projection

$$a \in l^\infty \mapsto \dot{a} \in l^\infty/c_0.$$

Notice that the equivalence class of a given sequence a depends only on the projection of a to l^∞/c_0.

It is not very hard to show that the norm of $\dot{a}$ is given by the asymptotic expression

$$\|\dot{a}\| = \lim_{n\to\infty} \|\{a_n, a_{n+1}, \ldots\}\|. \tag{1.9}$$

Let us now consider invariants for the equivalence relation $\sim$. One such invariant is the number $\|\dot{a}\|$, since $a \sim b$ implies $\|\dot{a}\| = \|\dot{b}\|$. Another invariant is the *essential spectrum*, defined for an element $a \in l^\infty$ as the spectrum of $\dot{a}$ in l^∞/c_0. It is an interesting exercise to prove the following description of the essential spectrum of a sequence as the set of all its cluster points:

PROPOSITION 1.10. *For every* $a \in l^\infty$,

$$\mathrm{sp}(\dot{a}) = \bigcap_{n=1}^{\infty} \overline{\{a_k : k \geqslant n\}}.$$

The following theorem is due to von Neumann, and asserts that the essential spectrum is a complete invariant for the equivalence relation $\sim$.

THEOREM 1.11. $a \sim b$ *if and only if*, $\mathrm{sp}(\dot{a}) = \mathrm{sp}(\dot{b})$.

A proof of the interesting implication can be found in [**19**, pp. 81–82].

We move now from commutative asymptotics to noncommutative asymptotics. In place of l^∞ we consider the space $\mathscr{L}$ of all infinite matrices A of complex numbers $A = (a_{ij})_{i,j\geqslant 1}$, for which the norm

$$\|A\| = \sup \left| \sum_{i,j=1}^{\infty} a_{ij}\xi_j\bar{\eta}_i \right|$$

is finite, the supremum extended over all sequences ξ, η of complex numbers such that $\xi_i = \eta_i = 0$ for all but a finite number of indices i and which satisfy

$$\sum_i |\xi_i|^2 \leqslant 1, \qquad \sum_i |\eta_i|^2 \leqslant 1.$$

$\mathscr{L}$ is a C^*-algebra relative to the usual matrix operations, and the usual $*$-operation in which A^* means the conjugate transpose of A. Of course, $\mathscr{L}$ is isometrically $*$-isomorphic to the algebra $\mathscr{L}(\mathscr{H})$ of all bounded operators on an infinite-dimensional separable Hilbert space $\mathscr{H}$, but for the moment, we wish to view $\mathscr{L}$ as a noncommutative analogue of the C^*-algebra of sequences l^∞.

Let $\mathscr{K}$ be the norm-closure in $\mathscr{L}$ of all matrices $A = (a_{ij})$ for which $a_{ij} = 0$ for all but a finite number of pairs i, j. $\mathscr{K}$ is a closed two-sided ideal in $\mathscr{L}$, occupying the role of the compact operators in $\mathcal{L}(\mathcal{H})$. Thus $\mathcal{L}/\mathcal{K}$ is a C^*-algebra, and we have a natural projection $A \in \mathcal{L} \mapsto \dot{A} \in \mathcal{L}/\mathcal{K}$. In analogy with (1.9), the quotient norm in $\mathcal{L}/\mathcal{K}$ is given by

$$\|\dot{A}\| = \lim_{n\to\infty} \|A_n\|, \qquad A \in \mathcal{L}, \tag{1.12}$$

where $A_1, A_2, \ldots$ is the sequence of truncated matrices

$$A_n = \begin{bmatrix} a_{n,n} & a_{n,n+1} & \cdots \\ a_{n+1,n} & a_{n+1,n+1} & \cdots \\ \vdots & \vdots & \end{bmatrix}.$$

Now every *-automorphism of $\mathcal{L}$ is implemented by a unitary matrix U in the sense that $\mathcal{Q}(A) = UAU^{-1}$, $A \in \mathcal{L}$. Thus the following definition makes precise in this context the idea that two matrices A and B should have the same asymptotic properties.

DEFINITION 1.13. *$A \sim B$ iff there is a unitary matrix U such that $UAU^{-1} - B \in \mathcal{K}$.*

What about invariants for the relation $A \sim B$? As before, we have the essential norm defined by (1.12), and the essential spectrum of $A \in \mathcal{L}$, defined as the spectrum of $\dot{A}$ relative to the C^*-algebra $\mathcal{L}/\mathcal{K}$. This time, however, neither of these is a complete invariant. Indeed, the situation here is vastly more complicated than the corresponding situation for l^∞.

A new invariant, which has no commutative counterpart, has to do with the Fredholm index. In order to discuss this, let us discard the matrix interpretation and consider $\mathcal{L}$ (resp. $\mathcal{K}$) to be the algebra $\mathcal{L}(\mathcal{H})$ of all bounded (resp. compact) operators on a Hilbert space $\mathcal{H}$ of dimension $\aleph_0$. An operator A is said to be *semi-Fredholm* if its range $A\mathcal{H}$ is closed and *one* of the two subspaces

$$\ker A \quad \textit{and} \quad \operatorname{Coker} A = \ker A^*$$

is finite dimensional. In this case the *index* of A is defined as the number

$$\operatorname{index} A = \dim\ker A - \dim\ker A^*.$$

The index of A is an ordinary integer or is $\pm\infty$. It is a fact that the semi-Fredholm operators are stable under compact perturbations, and that the index of such an operator is similarly stable.

$$\operatorname{index} A = \operatorname{index}(A + K), \tag{1.14}$$

for every semi-Fredholm operator A and every compact operator K.

The formula (1.14) implies that the index is a new invariant for the equivalence relation $A \sim B$. Indeed, if $A \sim B$ and $A - \lambda 1$ is semi-Fredholm for some complex scalar λ, then $B - \lambda 1$ is also semi-Fredholm and moreover

$$\operatorname{index}(A - \lambda 1) = \operatorname{index}(B - \lambda 1). \tag{1.15}$$

With the help of (1.15), we can now give an example of two inequivalent operators A, B which have the same essential spectrum. Let A be the simple unilateral shift and let $B = A^*$. Then both A and B have the same essential spectrum (namely the unit circle), whereas both A and B are Fredholm operators whose Fredholm indices are, respectively, -1, and $+1$. It follows that A and B cannot be equivalent.

Let us return now to quasitriangular operators. In 1968, Halmos showed [**37**] that there exist operators which are *not* quasitriangular (the shift is one such). Douglas and Pearcy later clarified the situation somewhat by proving [**28**].

THEOREM 1.16. *If A is quasitriangular, then* $\operatorname{index}(A - \lambda 1) \geq 0$ *for every* $\lambda \in \mathbf{C}$ *for which $A - \lambda 1$ is semi-Fredholm.*

This implies Halmos' earlier result, because the shift is a Fredholm operator whose index is -1.

In a series of papers which contain a deep analysis of the spectral properties of operators, Apostol, Foias, and Voiculescu provided a remarkable converse to Theorem 1.16 [**3**, Corollary 5.5].

THEOREM 1.17. *If A is an operator such that* $\operatorname{index}(A - \lambda 1) \geq 0$ *for every* $\lambda \in \mathbf{C}$ *for which $A - \lambda 1$ is semi-Fredholm, then A is quasitriangular.*

Thus, the only obstruction to membership in the class $\mathcal{Q}\mathcal{T}$ is an index obstruction. Theorem 1.17 also has implications about invariant subspaces; it implies that every nonquasitriangular operator is intransitive. For by Theorem 1.17, such an operator A would have a scalar translate $A - \lambda 1$ of negative index, hence $A^* - \bar{\lambda} 1$ would have an eigenvector, hence A would have an invariant subspace of codimension one. So if there exists a transitive operator on a Hilbert space (and I personally believe that such operators do exist), then it *must* be quasitriangular.

Lecture 2. Triangular and Quasitriangular Operator Algebras

The first nonselfadjoint operator algebras to be considered seriously were the triangular algebras of Kadison and Singer [**42**]. These are the complex subalgebras $\mathscr{A}$ of $\mathscr{L}(\mathscr{H})$ such that

$$\mathscr{A} \cap \mathscr{A}^* \textit{ is a maximal abelian subalgebra of } \mathscr{L}(\mathscr{H}).$$

There are two extreme types: the *transitive* ones (i.e. lat $\mathscr{A} = \{0, 1\}$) and the *reflexive* ones (an algebra $\mathscr{A}$ is reflexive iff $\mathscr{A} = \operatorname{alg}\operatorname{lat}\mathscr{A}$). Actually, Kadison and Singer used the terminology *irreducible* and *hyperreducible*, but I shall not.

A *maximal triangular algebra* is a maximal element in the partially ordered set of all triangular subalgebras of $\mathscr{L}(\mathscr{H})$. A straightforward exercise with Zorn's lemma shows that every triangular algebra is included in a maximal triangular algebra.

It is easy to describe the maximal triangular algebras on $\mathscr{H}$ when $\mathscr{H}$ is finite dimensional. For that, choose an orthonormal basis $e_1, \ldots, e_n$ for $\mathscr{H}$ and let $\mathscr{A}$ be the algebra of all operators on $\mathscr{H}$ whose matrix relative to $\{e_j\}$ is upper triangular. Then $\mathscr{A}$ is maximal triangular. Moreover, every maximal triangular algebra on $\mathscr{H}$ is unitarily equivalent to this particular one [**42**]. More generally, any subalgebra of $\mathscr{A}$ *which contains all diagonal matrices* is a triangular algebra; and the preceding discussion implies that every triangular subalgebra of $\mathscr{L}(\mathscr{H})$ is unitarily equivalent to such a subalgebra of $\mathscr{A}$. In particular, *there exist no transitive triangular algebras on a finite-dimensional Hilbert space.*

In infinite dimensions, the situation is quite different, and some examples will be described presently. I will not have very much to say about transitive triangular algebras in these lectures, but I do want to state some problems about them and review briefly some of the things we have learned.

The first problem asks if there exist any nontrivial transitive algebras at all (triangular or not) which are closed in the weak operator topology. Here, the term nontrivial means the obvious thing: $\mathscr{A} \neq \mathscr{L}(\mathscr{H})$. This is, of course, an algebraic and easier variant of the invariant subspace problem, and it is still unsolved. It has been known for some time that if a weakly closed transitive algebra $\mathscr{A} \subseteq \mathscr{L}(\mathscr{H})$ contains a maximal abelian von Neumann algebra, then $\mathscr{A} = \mathscr{L}(\mathscr{H})$ [**5**]. In particular, a transitive triangular algebra is always weakly dense in $\mathscr{L}(\mathscr{H})$.

Radjavi, Rosenthal, Nordgren, and others have found interesting generalizations of these results [**59**]. It is significant that transitive algebras exist which are not dense in $\mathscr{L}(\mathscr{H})$ relative to the σ-weak topology; indeed, Loebl and Muhly [**49**] have given an example of a proper σ-weakly closed subalgebra of $\mathscr{L}(\mathscr{H})$ which is weakly dense in $\mathscr{L}(\mathscr{H})$.

Now one of the reasons Kadison and Singer introduced triangular algebras was their hope that the *maximal* ones (even the transitive maximal ones) might play a role somewhat analogous to that of the upper triangular $n \times n$ matrices. For instance, Kadison has asked if every operator is a member of some triangular operator algebra, transitive or not. If the answer is yes, then one would have achieved something like a triangular form for the operator, even though one could draw no conclusions about the existence of invariant subspaces. This problem is still open, and still hard to approach.

The first examples of transitive triangular algebras appeared in [**42**], and can be described as follows. Let $\mathscr{M}$ be a maximal abelian von Neumann algebra in $\mathscr{L}(\mathscr{H})$ and let U be a unitary operator such that

(2.1)
(i) $U\mathscr{M}U^* = \mathscr{M}$, *and*
(ii) (*ergodicity*) *if* $E \in \mathscr{M}$ *is a projection such that* $UEU^* \leqslant E$, *then* $E = 0$ *or* 1.

For example, one may start with an ergodic measure preserving transformation ϕ of a *finite* measure space (X, μ), take $\mathscr{H} = L^2(X, \mu)$, $\mathscr{M} =$ the multiplication algebra of (X, μ), and $Uf(x) = f(\phi x)$, for $f \in L^2(x, \mu)$. The conditions (2.1) imply that the linear space $\mathscr{A}(\mathscr{M}, U)$ consisting of all "polynomials",

$$D_0 + D_1U + \cdots + D_nU^n,$$

$n = 0, 1, \ldots,$ $D_i \in \mathscr{M}$, is a transitive triangular algebra. However, these algebras $\mathscr{A}(\mathscr{M}, U)$ are not closed in any reasonable topology, and that is a feature most analysts find disagreeable. The preceding discussion implies that the weak operator closure of $\mathscr{A}(\mathscr{M}, U)$ is the trivial algebra $\mathscr{L}(\mathscr{H})$, and later work which will be described in Lectures 5 and 6 implies that even the σ-weak closure of $\mathscr{A}(\mathscr{M}, U)$ is $\mathscr{L}(\mathscr{H})$.

Fortunately, the *norm* closure of $\mathscr{A}(\mathscr{M}, U)$ is nontrivial, and in fact is itself a triangular algebra (see [**6**, Lemma 1.6]). These norm-closed triangular algebras $\mathscr{A}(\mathscr{M}, U)^{-\|\cdot\|}$ completely determine the action of $\mathbf{Z}$ on $\mathscr{M}$ (via $n \in \mathbf{Z} \mapsto \mathrm{ad}_{U^n}$) in that two algebras are isomorphic if and only if, the corresponding actions are conjugate. A version of this appears in [**6**] (with unitary equivalence replacing algebraic isomorphism), and the general result is in [**15**].

Recently, more information about transitive triangular algebras has been obtained by Baruch Solel [**69**]. Nevertheless, these algebras are still not very well understood. In particular, no one has a clear idea of how well the above examples represent general transitive triangular algebras. As a test question, we propose the following: Is the norm closure of a transitive triangular algebra also triangular?