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Second Microlocalization and Semilinear Caustics

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Foreword

This text grew up from lectures given at the University of Rennes I during the academic year 1988–1989. The main topics covered are second microlocalization along a lagrangian manifold, defined by Sjöstrand in [Sj], and its application to the study of conormal singularities for solutions of semilinear hyperbolic partial differential equations, developed by Lebeau [L4].

To give a quite self-contained treatment of these questions, we included some developments about FBI transformations and subanalytic geometry. The text is made of four chapters. In the first one, we define the Fourier-Bros-Iagolnitzer transformation and study its main properties. The second chapter deals with second microlocalization along a lagrangian submanifold, and with upper bounds for the wave front set of traces one may obtain using it. The third chapter is devoted to formulas giving geometric upper bounds for the analytic wave front set and for the second microsupport of boundary values of ramified functions. Lastly, the fourth chapter applies the preceding methods to the derivation of theorems about the location of microlocal singularities of solutions of semilinear wave equations with conormal data, in general geometrical situation. Every chapter begins with a short abstract of its contents, where are collected the bibliographical references.

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Main notations

TM = tangent bundle to the manifold M.
T_xM = fiber of TM at the point x of M.
T^*M = cotangent bundle to the manifold M.
T^*_xM = fiber of T^*M at the point x of M.
T_NM = normal bundle to the submanifold N of M.
T^*_NM = conormal bundle to the submanifold N of M.
For E a vector bundle over M, $E \setminus \{0\}$ or $E \setminus 0$ denotes E minus its zero section.
For E, F two fiber bundles over M, $E \times_M F$ denotes the fibered product of E by F over M.
Over a coordinate patch of M, $E \times_M F = \{ (x, e, f);\ e \in E_x,\ f \in F_x \}$.
If $h : M_1 \to M_2$ is a diffeomorphism between two manifolds, one denotes by $\tilde{h}$ the map it induces $\tilde{h} : T^*M_1 \to T^*M_2$. In local coordinates $\tilde{h}(x, \xi) = (h(x), {}^t dh(x)^{-1} \cdot \xi)$.
If $x_0 \in M_1$ and $y_0 \in M_2$, one denotes by $h : (M_1, x_0) \to (M_2, y_0)$ a germ of map from the germ of M_1 at x_0 to the germ of M_2 of y_0.
$\mathrm{gr}(\psi)$ = graph of a map ψ from a manifold to a manifold.
$d(\ , \ )$ = euclidean (resp. hermitian) distance on the real euclidean (resp. the complex hermitian) space.
$d(\ , L)$ = distance to a subset L.
d = exterior differential on a real manifold.
∂ = holomorphic differential on a complex analytic manifold.
$\bar{\partial}$ = antiholomorphic differential on a complex analytic manifold.
$dL(x)$ = Lebesgue measure on $\mathbb{C}^n$.
We will use the standard notation for the different spaces of distributions: C_0^∞ (compactly supported smooth functions), S (Schwartz space), S' (tempered distributions), H^s (Sobolev spaces), ...

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0. Introduction

We will first recall some elementary results concerning the Cauchy problem for the linear wave equation. Then, we will indicate the new phenomenons appearing in the study of semilinear wave equations and we will describe the theorems obtained by Beals, Bony, Melrose-Ritter about semilinear Cauchy problems with conormal data. Lastly, we will state "swallow-tail's problem", which will be solved in the last chapter of this text, where we expose a method due to Lebeau.

Let us consider on $\mathbb{R}^{1+d}$ with coordinates $(t,x) = (t, x_1, \dots, x_d)$ the wave operator

$$\square = \frac{\partial^2}{\partial t^2} - \Delta_x = \frac{\partial^2}{\partial t^2} - \sum_{j=1}^{d} \frac{\partial^2}{\partial x_j^2} \,. \tag{1}$$

To solve the Cauchy problem is to find a solution $u(t,x)$ to the problem

$$\begin{aligned} \square u &= f(t,x) \qquad t>0 \\ u|_{t=0} &= u_0(x) \\ \left.\frac{\partial u}{\partial t}\right|_{t=0} &= u_1(x) \end{aligned} \tag{2}$$

where the functions f, u_0, u_1 are given in convenient spaces.

Let us first consider the special case $f \equiv 0$, $u_0 \equiv 0$, $u_1 = \delta$, Dirac mass at the origin of $\mathbb{R}^d$. Using a Fourier transformation with respect to x, one sees that (2) admits a unique solution $e_+(t,x)$ in the space of continuous functions of $t \in \bar{\mathbb{R}}_+$ with values in the space of tempered distributions on $\mathbb{R}^d$, whose Fourier transform with respect to x is given by

$$(\mathcal{F}_x e_+)(t,\xi) = \frac{\sin t|\xi|}{|\xi|} \mathbf{1}_{\{t\geq 0\}} \,. \tag{3}$$

It follows from the preceding expression and from the Paley-Wiener theorem that $e_+(t,x)$ is supported inside the forward solid light cone $\tilde{\Gamma} = \{ (t,x);\ |x| \leq t \}$.

The elementary solution $e_+(t,x)$ allows us to solve in general problem (2):

Theorem 1. *Let $f \in L^\infty(\mathbb{R}_+, H^{s-1}(\mathbb{R}^d))$, $u_0 \in H^s(\mathbb{R}^d)$, $u_1 \in H^{s-1}(\mathbb{R}^d)$. Then (2) has a unique solution $u \in C^1(\bar{\mathbb{R}}_+, S'(\mathbb{R}^d))$. It is given by*

$$\begin{aligned} u(t,x) = \int_0^t \int e_+(t-s, x-y) f(s,y)\, ds\, dy + e_+ * [u_0 \otimes \delta'_{t=0}] \\ + e_+ * [u_1 \otimes \delta_{t=0}] \,. \end{aligned} \tag{4}$$

Proof. Let us remark first that because of the support properties of e_+, the convolutions make sense. One then checks at once that the function u given by (4) is a solution of (2), and satisfies, because of (3), the regularity conditions given in the statement of the theorem. The assertion of uniqueness is trivial.

One should remark that it follows from (4), and from expression (3), that if for every $k \in \mathbb{N}$ $D_t^k f \in L^2(\bar{\mathbb{R}}_+, H^{s-1-k}(\mathbb{R}^d))$, then $D_t^k u \in L^2(\bar{\mathbb{R}}_+, H^{s-k}(\mathbb{R}^d))$. This implies that $u|_{t>0}$ is in the space $H^s_{\mathrm{loc}}(\bar{\mathbb{R}}^{1+d}_+)$ if $f \in H^{s-1}_{\mathrm{loc}}(\bar{\mathbb{R}}^{1+d})$. In fact, one has just to write with $k = [s] + 1$

$$\int \hat{u}(\tau,\xi)^2(1+\xi^2+\tau^2)^s\, d\xi\, d\tau \le \int_{|\tau|\le|\xi|} \hat{u}(\tau,\xi)^2(1+\xi^2)^s\, d\xi\, d\tau + \int_{|\xi|\le|\tau|} \hat{u}(\tau,\xi)^2(1+\xi^2)^{s-k}(1+\tau^2)^k\, d\xi\, d\tau\ .$$

The formula (4) shows that the value of u at (t,x) depends just on the value of f at points belonging to $(t,x) - \tilde{\Gamma}$ and on the value of u_0, u_1 at points of $\{\, y \in \mathbb{R}^d;\ (0,y) \in (t,x) - \tilde{\Gamma}\,\}$ (finite propagation speed). If Ω is an open subset of $\mathbb{R}^{1+d}$, one says that Ω is a determination domain of $\omega = \Omega \cap \{t=0\}$ if and only if for every $(t,x) \in \Omega$, the set

$$\{\,(s,y);\ (\operatorname{sgn} t)(t-s) > |x-y| \text{ and } (\operatorname{sgn} t)(\operatorname{sgn} s) \ge 0\,\}$$

is contained in Ω. Using convenient cut-off functions, one deduces from Theorem 1 and from the finite propagation speed property:

Theorem 2. *Let Ω be a determination domain of ω. Let $u_0 \in H^s_{\mathrm{loc}}(\omega)$, $u_1 \in H^{s-1}_{\mathrm{loc}}(\omega)$ and let f be a distribution on Ω which is, locally in Ω, in the space $L^\infty(\mathbb{R}, H^{s-1}(\mathbb{R}^d))$. Then the problem*

$$\text{(5)}\qquad \begin{aligned} \Box u &= f(t,x) && \text{in } \Omega \\ u|_{t=0} &= u_0 && \text{on } \omega \\ \left.\frac{\partial u}{\partial t}\right|_{t=0} &= u_1 && \text{on } \omega \end{aligned}$$

has a unique solution u which is in $C^0(\mathbb{R}, H^{s-1}(\mathbb{R}^d))$ locally in Ω. Moreover u belongs to $H^s_{\mathrm{loc}}(\Omega)$ if $f \in H^{s-1}_{\mathrm{loc}}(\Omega)$.

Let us now recall the theorem of propagation of C^∞ microlocal singularities. We will use the notion of C^∞ wave front set, whose definition is recalled in Section 1 of Chapter I. Let us denote by $\operatorname{Car}\Box = \{(t,x;\tau,\xi) \in T^*\Omega;\ \xi^2 = \tau^2\,\}$ the characteristic variety of $\Box$. If A is a subset of $T^*\Omega \cap \{\pm t \ge 0\}$, one will denote by $\mathcal{P}_+(A)$ (resp. $\mathcal{P}_-(A)$) the union of A and of the forward (resp. backward) integral curves of the hamiltonian field of $\sigma(\Box) = \xi^2 - \tau^2$ issued from the points of $A \cap \operatorname{Car}\Box$, and contained in Ω:

$$\text{(6)}\quad \mathcal{P}_\pm(A) = A \cup \Big(\{\,(t,x;\tau,\xi);\ \pm t > 0,\ \xi^2 = \tau^2 \text{ and there is } s \in \mathbb{R} \text{ with } \pm s\tau < 0,\ (t+s\tau, x-s\xi;\tau,\xi) \in A\,\} \cap T^*\Omega\Big)\ .$$

Since Ω is a determination domain, as soon as there is $(t,x;\tau,\xi) \in \mathcal{P}_\pm(A)$ with $\xi^2 = \tau^2$ and $s_0 \in \mathbb{R}$ such that $(t+s_0\tau, x-s_0\xi;\tau,\xi) \in \mathcal{P}_\pm(A)$, then the points $(t+s\tau, x-s\xi;\tau,\xi)$ belong to $\mathcal{P}_\pm(A)$ for every $s \in [0,s_0]$.

The theorem of propagation of microlocal singularities is then:

Theorem 3. *Let u be a solution on Ω of the Cauchy problem* (5). *One has*

$$\mathrm{WF}(u)|_{\pm t>0} \subset \mathcal{P}_\pm[(\mathrm{WF}(f) \cap \{\pm t > 0\}) \cup \{ (0,x;\tau,\xi);\ \xi^2 = \tau^2 \text{ and } (x,\xi) \in \mathrm{WF}(u_0) \cup \mathrm{WF}(u_1) \}] . \tag{7}$$

Proof. One knows (see [H], Section 8.2) that if v_1 and v_2 are two compactly supported distributions

$$\mathrm{WF}(v_1 * v_2) \subset \{ (z,\zeta);\ \exists(z_1,z_2) \text{ with } (z_1,\zeta) \in \mathrm{WF}(v_1), (z_2,\zeta) \in \mathrm{WF}(v_2) \text{ and } z = z_1 + z_2 \} . \tag{8}$$

Because of (4), we thus see that the inclusion (7) follows from the following lemma:

Lemma 4. *One has*

$$\mathrm{WF}(e_+) \subset T^*_{\{0\}}\mathbb{R}^{1+d} \cup \{ (t,x;\tau,\xi);\ t>0,\ t^2 = x^2, (\tau,\xi) = \lambda(t,-x) \text{ with } \lambda \in \mathbb{R} \} . \tag{9}$$

Proof. To show (9) we will prove that e_+ is conormal along the forward light cone. More precisely, let $\mathcal{M}$ be the $C^\infty(\mathbb{R}^d)$-module of C^∞ vector fields whose symbol vanishes on the right hand side of (9). We will show that if $(X_1,\dots,X_m)$ is an m-tuple of elements of $\mathcal{M}$ one has $X_1\cdots X_m e_- \in H^\sigma_{\mathrm{loc}}(\mathbb{R}^{1+d})$ for every $\sigma < \frac{1-d}{2}$. One sees easily that $\mathcal{M}$ is generated by the fields

$$\begin{aligned} & t\frac{\partial}{\partial t} + \sum_1^d x_j \frac{\partial}{\partial x_j} && \\ & x_j\frac{\partial}{\partial x_k} - x_k\frac{\partial}{\partial x_j} && 1 \le j \ne k \le d \\ & x_j\frac{\partial}{\partial t} + t\frac{\partial}{\partial x_j} && 1 \le j \le d. \end{aligned} \tag{10}$$

The action of the first one on e_+ gives $-(d-1)e_+$ and the other ones cancel e_+. For every compactly supported χ and every m-tuple of vector fields $X_1,\dots,X_m$ of the form (10) one has thus

$$|\chi(\widehat{X_1\cdots X_m e_+})| \le C_m|\widehat{\chi e_+}(\tau,\xi)| \le \frac{C'_m}{1+|\tau|+|\xi|} \tag{11}$$

where the last inequality follows from (3).

Let $\Gamma = \{ (t,x);\ t = |x| \}$. The inclusion (9) may now be deduced from (11) in the following way: if $(t_0,x_0) \in \mathrm{Supp}(e_+)$, $t_0^2 \ne x_0^2$, the fields $\chi(x)\frac{\partial}{\partial x_j}$, $j = 1,\dots,d$, and

$\chi(x)\frac{\partial}{\partial t}$ with $\chi \in C_0^\infty(\mathbb{R}^{1+d})$, $\operatorname{Supp}\chi \cap \Gamma = \emptyset$, are in $\mathcal{M}$ and thus e_+ is C^∞ close to (t_0, x_0). On the other hand, if (t_0, x_0) satisfies $t_0^2 = x_0^2 \neq 0$, there is, close to (t_0, x_0), a system of local coordinates $(y^0, \dots, y^d)$ such that Γ is given by $y^0 = 0$. Then, the fields $\chi(y)\frac{\partial}{\partial y^1}$, $\dots$, $\chi(y)\frac{\partial}{\partial y^d}$ are in $\mathcal{M}$ if $\operatorname{Supp}\chi$ is small enough. It follows that $\mathrm{WF}(e_+) \subset T^*_\Gamma \mathbb{R}^{1+d}$ close to (t_0, x_0).

In the preceding proof, we used the upper bound (11) of $|\widehat{\chi e_+}|$. In fact, there is a better upper bound, we will have to use in Chapter IV:

Lemma 5. *For every $\chi \in C_0^\infty(\mathbb{R}^{1+d})$ there is a constant $C > 0$ with*

$$|\widehat{\chi e_+}(\tau, \xi)| \leq C(1 + |\xi| + |\tau|)^{-1}(1 + ||\xi| - |\tau||)^{-1} . \tag{12}$$

Proof. Because of the support property enjoyed by e_+, we may always assume that χ is a compactly supported function of the single variable t. Then, by (3), $\widehat{\chi e_+}(\tau, \xi) = \int_0^{+\infty} e^{-it\tau}\chi(t)\frac{\sin t|\xi|}{|\xi|}\, dt$. Using that for any complex number α one has

$$\left| \int_0^{+\infty} \chi(t) e^{-it\alpha}\, dt \right| \leq C(1 + |\alpha|)^{-1}$$

the inequality (12) follows.

Before beginning the description of the nonlinear problems we will be interested in, let us mention that, of course, Theorem 3 admits a more precise statement. In fact, as is well known (see [H]), $\mathrm{WF}(u)\backslash\mathrm{WF}(f)$ is foliated by the integral curves of the hamiltonian field of $\sigma(\Box)$.

We will now study the problem of control of microlocal singularities of the solution u, given in the space $H^s_{\mathrm{loc}}(\Omega)$ with $s > \frac{1+d}{2}$, of a semilinear Cauchy problem of the form

$$\begin{aligned} \Box u &= f(t, x, u) \\ u|_{t=0} &= u_0 \\ \left.\frac{\partial u}{\partial t}\right|_{t=0} &= u_1 \end{aligned} \tag{13}$$

where f is a C^∞ function over $\mathbb{R}^{1+d} \times \mathbb{R}$ and u_0, u_1 are given on $\omega = \Omega \cap \{t = 0\}$ in the space $H^s_{\mathrm{loc}}(\omega)$ and $H^{s-1}_{\mathrm{loc}}(\omega)$ respectively.

The new phenomenon one has to cope with to solve such a problem, is the one of interaction of singularities. For instance, let us take two distributions with compact support on $\mathbb{R}^n$ v_1, v_2 and assume that $\mathrm{WF}(v_j) \subset \{ (0; \lambda\xi^j),\ \lambda \geq 0 \}$, where ξ^1 and ξ^2 are two non-zero elements of $T^*_0\mathbb{R}^n$ such that there exist no negative real number ϱ with $\xi^1 = \varrho\xi^2$. Assume moreover that v_1 and v_2 belong to $H^\sigma(\mathbb{R}^n)$ for some $\sigma > n/2$. Then the product $v_1 v_2$ exists, and defines an element of $H^\sigma(\mathbb{R}^n)$. Writing $\widehat{v_1 \cdot v_2}(\xi) = \hat{v}_1 * \hat{v}_2(\xi)$, one sees easily that

$$\mathrm{WF}(v_1 v_2) \subset \{ (0, \lambda_1\xi^1 + \lambda_2\xi^2);\ \lambda_1 \geq 0,\ \lambda_2 \geq 0 \} . \tag{14}$$

In general, there is no better upper bound for $\mathrm{WF}(v_1 v_2)$, i.e. there are, in this last set, directions belonging neither to $\mathrm{WF}(v_1)$ nor to $\mathrm{WF}(v_2)$. Moreover, if ξ^1 and ξ^2 belong to a same line and have opposite directions, the inclusion (14) is no longer true and any $\xi \in T_0^*\mathbb{R}^n$ may be inside $\mathrm{WF}(v_1 v_2)$.

A similar phenomenon happens when one computes $f(v)$ with f a C^∞ function of v. This suggest that, in general, the solution of a semilinear problem like (13) will have much more singularities than the solution of the linear problem (5). As a matter of fact, it is reasonable to suppose that u will have at least the singularities of the solution to the linear problem, i.e. those given by the right hand side of (7) with $f = 0$. But then, in the nonlinear term $f(t,x,u)$ of (12), these singularities will create new ones by interaction, that is $\mathrm{WF}(f(t,x,u))$ will be bigger than $\mathrm{WF}(u)$. By (7) the upper bound for $\mathrm{WF}(u)$ will have to take into account the points obtained by propagation from $\mathrm{WF}(f(t,x,u)) \cap \mathrm{Car}\,\Box$. These new singularities will also, by interaction, contribute to increase $\mathrm{WF}(f(t,x,u))$ and so on.

In general, one cannot hope to obtain for nonlinear problems results like Theorem 3. In fact, Beals [Be1] found an example of a solution of a semilinear Cauchy problem, with Cauchy data smooth outside 0, and whose singularities are dense inside the light cone $\{(t,x);\ |x| \le t\}$. To get anyway results of control of singularities, one is thus lead to make specific assumptions on the nature of the singularity of the Cauchy data u_0, u_1. In particular, the notion of "conormal regularity" happened to be very well adapted to that. Let V be a submanifold of the hyperplane $\{t = 0\}$. One says that $u_j \in H_V^{s-j,+\infty}$ if for every integer m and for every m-tuple of C^∞ vector fields $X_1, \ldots, X_m$ tangent to V, one has $X_1 \cdots X_m u_j \in H^{s-j}_{\mathrm{loc}}$. In particular, $\mathrm{WF}(u_j)$ is contained inside $T_V^*\mathbb{R}^d$. If one solves a linear problem like (5) with $f = 0$ and such initial data, it follows from Theorem 3 that

$$\text{(15)} \qquad \mathrm{WF}(u)|_{t>0} \subset \left\{ (t,x;\tau,\xi);\ t>0,\ \xi^2 = \tau^2 \neq 0,\ (x - t\frac{\xi}{\tau}, \xi) \in T_V^*\mathbb{R}^d \right\}.$$

When V is a hypersurface, the projection of the right hand side of (15) on $\mathbb{R}^{1+d}$ is close to $t = 0$ the union of two smooth hypersurfaces intersecting transversally along V. In the case of a semilinear Cauchy problem like (13), Bony [Bo1], [Bo3] proved that the inclusion (15) remains valid for t close to 0. In fact, the solution u is conormal along the two outgoing hypersurfaces.

This result thus shows that close to $t = 0$, the solution of the semilinear problem has the same singularities as the solution of the linear one. Anyway, on a longer interval of time, other singularities happen as a consequence of nonlinear interaction. Let us consider in 2 space dimension a solution $u \in H^s_{\mathrm{loc}}$, with $s > \frac{d+1}{2}$, of the equation $\Box u = f(t,x,u)$, such that $u|_{t<t_0<0}$ is conormal along three characteristic hypersurfaces Σ_1, Σ_2, Σ_3 which, in $t < t_0$, intersect just two by two and transversally (conormal still meaning that $u|_{t<t_0}$ keeps a fixed Sobolev regularity when one applies to it any number of C^∞ vector fields tangent to $\Sigma_1 \cup \Sigma_2 \cup \Sigma_3$). Assume moreover that in $\{t > t_0\}$, Σ_1, Σ_2, Σ_3 intersect transversally at a single point 0. Then, it has been proved independently by Bony [Bo2] and Melrose-Ritter [M-R] that the solution u is C^∞ outside $\Sigma_1 \cup \Sigma_2 \cup \Sigma_3 \cup \Gamma$ where Γ is the boundary of the forward light cone with vertex at 0, and that u is conormal along the smooth points of this intersection (see also Chemin [Ch] for an extension and Beals [Be2], [Be3] for a more elementary proof). In such a case, we thus see that interaction of singularities provokes the creation of new singularities along Γ.

The fourth Chapter of this text will be devoted to the study of a phenomenon of interaction of singularities in the large. Consider in $d = 2$ space dimension a solution u of a semilinear wave equation, whose Cauchy data are conormal along a real analytic curve V of $\mathbb{R}^2$, having at a single point a non-degenerate minimum of its curvature radius (for instance, a parabola).

The projection on $\mathbb{R}^2$ of the flow out of $T^*_V\mathbb{R}^3 \cap \mathrm{Car}\,\square$ by the hamiltonial field is the union of two hypersurfaces of $\mathbb{R}^3$, which are smooth close to $t = 0$, V_+, and V_-. One of them, say V_-, remains smooth in the future, but the other one, V_+, has a pinching point in $t > 0$ (V_+ is a swallow tail). The aim of Chapter IV is to prove, following Lebeau [L4], that $u|_{t>0}$ is smooth outside the union of V_-, V_+ and of the two-dimensional forward light cone with vertex at the pinching point of V_+.

I. Fourier-Bros-Iagolnitzer transformation and first microlocalization

This first chapter is devoted to the definition of Fourier-Bros-Iagolnitzer (FBI) transformation and to its application to the study of microlocal regularity of distributions. The first section studies FBI transformations with quadratic phases, as those introduced by Bros-Iagolnitzer [Br-I] and Sjöstrand [Sj]. In particular, we prove a characterization, due to P. Gérard [G], of H^s microlocal regularity of distributions in terms of FBI transformations. We also give, following [H], an inversion formula due to Lebeau [L1], expressing a distribution as an integral of its FBI transform.

In the second section, we bring out the fundamental properties enjoyed by the quadratic phase $\frac{i(x-t)^2}{2}$. This enables us to define general FBI transformations, using phases satisfying these properties. We still follow the bibliographical reference [Sj].

The third section gives the definition of Sjöstrand's spaces and of transformations between these spaces given by convenient phase integrals. We introduce the notion of "good contour" and prove the "fundamental lemma" of [Sj].

The last section is intended for a proof of the theorem of change of FBI: following Sjöstrand, we show that one may pass from a FBI defined by a phase g to a FBI defined by a phase $\tilde{g}$ using one of the transformations studied in the third section. This allows us to deduce from the results of Section 1 a characterization of microlocal H^s regularity in terms of FBI transformations with general phases.

1. FBI transformation with quadratic phase

Let u be a compactly supported distribution on $\mathbb{R}^n$. The FBI transformation of u is the function on $\mathbb{C}^n \times [0, +\infty[$ defined by:

$$Tu(x, \lambda) = \int e^{-\frac{\lambda}{2}(x-t)^2} u(t)\, dt \ . \tag{1.1}$$

It is an entire function of the complex variable x, real analytic with respect to the parameter λ. As u is of finite order, there exists an integer N and a constant $C > 0$ such that

$$|Tu(x, y)| \leq C(1 + \lambda + |\operatorname{Im} x|)^N e^{\frac{\lambda}{2}(\operatorname{Im} x)^2} \tag{1.2}$$

for $x \in \mathbb{C}^n$, $\lambda \in [0, +\infty[$.

The transformation (1.1) is nothing else than a modified Fourier transform. As this one, it will allow us to characterize (microlocal) regularity of u through better estimates than (1.2), the great parameter λ playing now the same role than the norm of the frequency variable in usual Fourier transform. Let us begin by the study of Sobolev regularity. Recall the following:

Definition 1.1. Let u be a distribution on $\mathbb{R}^n$. One says that u is H^s *microlocally at* $(t_0, \tau_0) \in T^*\mathbb{R}^n - \{0\}$ (what will be denoted by $u \in H^s_{(t_0,\tau_0)}$) or that (t_0, τ_0) *is not in the* H^s*-wave front set of* u ($(t_0, \tau_0) \notin \mathrm{WF}_s(u)$) if there is $\chi \in C_0^\infty(\mathbb{R}^n)$, $\chi \equiv 1$ close to t_0, and Γ a conic neighborhood of τ_0 in $\mathbb{R}^n - \{0\}$ such that

$$\int_\Gamma \langle\tau\rangle^s |\widehat{\chi u}(\tau)|^2 \, d\tau < +\infty \tag{1.3}$$

where $\langle\tau\rangle^2 = 1 + \tau^2$.

Our first aim is to prove, following P. Gérard [G], that we may characterize the preceding H^s-wave front set using Tu. Assuming u compactly supported – which does not restrict the generality of the problem – we have:

Theorem 1.2 (P. Gérard). *The point* $(t_0, \tau_0) \in T^*\mathbb{R}^n - \{0\}$ *is not in* $\mathrm{WF}_s(u)$ *if and only if there exists* W *neighborhood of* $x_0 = t_0 - i\tau_0$ *in* $\mathbb{C}^n$ *such that*

$$\int_1^{+\infty} \lambda^{\frac{3n}{2}+2s-1} \int_W |Tu(x,\lambda)|^2 e^{-\lambda(\mathrm{Im}\, x)^2} \, dL(x) \, d\lambda < +\infty \, , \tag{1.4}$$

$dL(x)$ *standing for Lebesgue's measure on* $\mathbb{C}^n$.

One should remark that, because of (1.2), one could replace in the first integral of (1.4) the lower bound 1 by any real positive number without changing the condition. The proof of the theorem relies on the following lemma.

Lemma 1.3. *For a compactly supported distribution* $u \in \mathcal{S}'(\mathbb{R}^n)$, *let us put*

$$\Sigma_s(u) = \Big\{ \tau_0 \in \mathbb{R}^n \setminus \{0\};\ \forall\Gamma \text{ open conic neighborhood of } \tau_0 \quad \int_\Gamma \langle\tau\rangle^{2s} |\hat{u}(\tau)|^2 \, d\tau = +\infty \Big\} \, .$$

A point $\tau_0 \in \mathbb{R}^n - \{0\}$ *is not in* $\Sigma_s(u)$ *if and only if there exists a neighborhood* V *of* τ_0 *in* $\mathbb{R}^n$ *such that*

$$\int_1^{+\infty} \lambda^{\frac{3n}{2}+2s-1} \int_{\mathbb{R}^n - iV} |Tu(x,\lambda)|^2 e^{-\lambda(\mathrm{Im}\, x)^2} \, dL(x) \, d\lambda < +\infty \, . \tag{1.5}$$

Proof. Let $\tilde{T}u(x, \lambda)$ be

$$\tilde{T}u(x,\lambda) = e^{-\frac{\lambda}{2}(\mathrm{Im}\, x)^2} Tu(x,\lambda) = \int e^{-\frac{\lambda}{2}(\mathrm{Re}\, x - t)^2 - i\lambda\, \mathrm{Im}\, x(\mathrm{Re}\, x - t)} u(t) \, dt \, . \tag{1.6}$$

The Fourier transform of $\tilde{T}u(s - i\tau, \lambda)$ $(s, \tau \in \mathbb{R}^n)$ with respect to s is

$$\widehat{\widetilde{T}}u(\sigma,\tau,\lambda)=\left(\frac{2\pi}{\lambda}\right)^{\frac{n}{2}}e^{-\frac{\lambda}{2}(\tau-\frac{\sigma}{\lambda})^2}\hat{u}(\sigma) \tag{1.7}$$

whence the equality

$$\int_{\mathbb{R}^n-iV}|Tu(x,\lambda)|^2e^{-\lambda(\operatorname{Im}x)^2}\,dL(x)=\left(\frac{2\pi}{\lambda}\right)^n\int_{\mathbb{R}^n}d\sigma\int_V d\tau e^{-\lambda(\tau-\frac{\sigma}{\lambda})^2}|\hat{u}(\sigma)|^2\ . \tag{1.8}$$

Since the contribution to the last integral coming from the domain $\{|\sigma|\leq 1\}$ is exponentially decreasing with respect to λ, it is enough, to show the lemma, to prove that there is a relatively compact neighborhood V of τ_0 in $\mathbb{R}^n-\{0\}$ such that

$$\int_1^{+\infty}\lambda^{\frac{n}{2}-1}\int_{|\sigma|\geq 1}K_V^s(\sigma,\lambda)|\hat{u}(\sigma)|^2\,d\sigma\,d\lambda<+\infty \tag{1.9}$$

with

$$K_V^s(\sigma,\lambda)=\lambda^{2s}\int_V e^{-\lambda(\tau-\frac{\sigma}{\lambda})^2}\,d\tau\ . \tag{1.10}$$

Let us show that if $V\subset\subset V'$ are two relatively compact open subsets of $\mathbb{R}^n-\{0\}$ and if $\Gamma=\bigcup_{t\geq 1}tV$, $\Gamma'=\bigcup_{t\geq 1}tV'$, there exist $C>0$ and $\varepsilon>0$ such that for every $\sigma\in\mathbb{R}^n$ with $|\sigma|\geq 1$ one has:

$$C^{-1}\lambda^{\frac{n}{2}}|\sigma|^{2s}\mathbf{1}_\Gamma(\sigma)\int_V e^{-(\lambda\tau-\sigma)^2}\,d\tau\leq K_{V'}^s(\sigma,\lambda) \tag{1.11}$$

$$K_V^s(\sigma,\lambda)\leq C\lambda^{\frac{n}{2}}|\sigma|^{2s}\mathbf{1}_{\Gamma'}(\sigma)\int_{V'}e^{-(\lambda\tau-\sigma)^2}\,d\tau+Ce^{-\varepsilon(\lambda+\frac{|\sigma|^2}{\lambda})}\ . \tag{1.12}$$

In fact, let V'' be an open subset such that $V\subset\subset V''\subset\subset V'$.

– If $\frac{\sigma}{\lambda}\notin V''$, one has $|\tau-\frac{\sigma}{\lambda}|\geq\varepsilon(1+\frac{|\sigma|}{\lambda})$ for every $\tau\in V$. Thus (1.12) is true because of the exponential term in its right hand side and (1.11) is trivial.

– If $\frac{\sigma}{\lambda}\in V''$, one has $\mathbf{1}_{\Gamma'}(\sigma)=1$ and $|\sigma|\sim\text{cst}\cdot\lambda$. Then, if r denotes the distance between $\overline{V}''$ and $\partial V'$, $r=d(\overline{V}'',\partial V')$

$$|\sigma|^{2s}\int_{V'}e^{-(\lambda\tau-\sigma)^2}\,d\tau\geq\text{cst}\cdot\lambda^{2s}\int_{|\tau|\leq r}e^{-\lambda^2|\tau|^2}\,d\tau\geq\text{cst}\cdot\lambda^{2s-n}$$

and on the other hand

$$K_V^s(\sigma,\lambda)\leq\lambda^{2s}\int_{\mathbb{R}^n}e^{-\lambda\tau^2}d\tau=\text{cst}\cdot\lambda^{2s-\frac{n}{2}}$$

whence the inequality (1.12). In the same way, since $\lambda\sim\text{cst}\cdot|\sigma|$,

$$K_{V'}^s(\sigma,\lambda)\geq\lambda^{2s}\int_{|\tau|\leq r}e^{-\lambda|\tau|^2}\,d\tau\geq\text{cst}\cdot|\sigma|^{2s}\lambda^{-\frac{n}{2}}$$

and

$$\int_V e^{-(\lambda\tau-\sigma)^2}\,d\tau\leq\text{cst}\cdot\lambda^{-n}$$

whence (1.11).

Modifying V if necessary, we deduce from (1.11) and (1.12) that (1.9) is equivalent to

$$\int_{\sigma\in\Gamma}\int_1^{+\infty}\lambda^{n-1}\,d\lambda\int_V e^{-(\lambda\tau-\sigma)^2}\,d\tau|\hat{u}(\sigma)|^2|\sigma|^{2s}\,d\sigma<+\infty \tag{1.13}$$

with $\Gamma=\bigcup_{t\geq 1}tV$.

One may always assume V of the form

$$V=\{\tau\in\gamma_0;\ \alpha<|\tau|<\beta\} \tag{1.14}$$

where γ_0 is an open cone in $\mathbb{R}^n-\{0\}$ and $\beta>\alpha>0$. One has then

$$\begin{aligned}\int_1^{+\infty}\lambda^{n-1}\,d\lambda\int_V e^{-(\lambda\tau-\sigma)^2}\,d\tau &= \int_1^{+\infty}\frac{d\lambda}{\lambda}\int_{\lambda V}e^{-(\tau-\sigma)^2}\,d\tau\\ &= \int_\Gamma e^{-(\tau-\sigma)^2}\Big[\int_1^{+\infty}\frac{d\lambda}{\lambda}\mathbf{1}_{\{\tau\in\lambda V\}}(\lambda)\Big]\,d\tau\\ &= \log\frac{\beta}{\alpha}\cdot\int_\Gamma e^{-(\tau-\sigma)^2}\,d\tau\,.\end{aligned}$$

The last integral is uniformly bounded from above when σ describes $\mathbb{R}^n$, and uniformly bounded from below by a positive constant when σ stays in Γ' with $\Gamma'\subset\subset\Gamma$.

It follows that (1.13) (and thus (1.9)) is equivalent (after a modification of Γ) to the condition

$$\int_{\sigma\in\Gamma}|\sigma|^{2s}|\hat{u}(\sigma)|^2\,d\sigma<+\infty \tag{1.15}$$

which is equivalent to $\tau_0\notin\Sigma_s(u)$. The lemma is proved.

Proof of Theorem 1.2: The distribution u is H^s microlocally at (t_0,τ_0) if and only if there exists $\chi\in C_0^\infty(\mathbb{R}^n)$, $\chi\equiv 1$ close to t_0, such that $\tau_0\notin\Sigma_s(\chi u)$ and thus, because of the lemma, such that there is a neighborhood V of τ_0 with

$$\int_1^{+\infty}\lambda^{\frac{3n}{2}+2s-1}\int_{\mathbb{R}^n-iV}|T(\chi u)(x,\lambda)|^2e^{-\lambda(\operatorname{Im}x)^2}\,dL(x)\,d\lambda<+\infty\,. \tag{1.16}$$

We just have to see that this condition is equivalent to (1.4).

Assume first that (1.16) is true and let U be a neighborhood of t_0 such that $\chi\equiv 1$ close to $\overline{U}$. The integral defined as (1.16) with the integration domain $\mathbb{R}^n-iV$ replaced by $U-iV$ is finite. But if $\operatorname{Re}x\in U$,

$$|(Tu-T(\chi u))(x,\lambda)|=|T((1-\chi)u)(x,\lambda)|\leq Ce^{\frac{\lambda}{2}((\operatorname{Im}x)^2-\varepsilon)}$$

and so (1.4) is satisfied with $W=U-iV$.

Suppose now that (1.4) is true. Let U be a neighborhood of t_0, V be a neighborhood of τ_0 such that $U-iV\subset\subset W$ and let $\chi\in C_0^\infty(U)$ with $\chi\equiv 1$ close to t_0. Writing

$$\chi(s)=\int e^{is\sigma}\hat{\chi}(\sigma)\frac{d\sigma}{(2\pi)^n}=(-\lambda)^n\int e^{i\lambda s(\tau-\sigma)}\hat{\chi}(\tau-\sigma)\frac{d\sigma}{(2\pi)^n}$$

we get, with the notation (1.6):

$$\tilde{T}(\chi u)(t-i\tau,\lambda) = (-\lambda)^n \int \tilde{T}u(t-i\sigma,\lambda)e^{i\lambda t(\tau-\sigma)}\hat{\chi}(\lambda(\tau-\sigma))\,\frac{d\sigma}{(2\pi)^n}\,.$$

Since $\hat{\chi}$ is rapidly decreasing, we obtain

$$\int_{U-iV} |\tilde{T}(\chi u)(x,\lambda)|^2\,dL(x) \le C\int_W |\tilde{T}u(x,\lambda)|^2\,dL(x) + O(\lambda^{-\infty})$$

whence (1.16).

Theorem 1.2 gives as a corollary a characterization of the C^∞ wave front set in terms of FBI transformation.Recall that the point $(t_0,\tau_0)\in T^*\mathbb{R}^n-\{0\}$ is not in the C^∞ wave front set of the distribution u, $\mathrm{WF}(u)$, if and only if there exist $\chi\in C_0^\infty(\mathbb{R}^n)$, $\chi\equiv 1$ close to t_0 and a conic neighborhood Γ of τ_0 in $\mathbb{R}^n-\{0\}$ such that for every integer N:

$$\sup_{\Gamma}\langle\tau\rangle^N|\widehat{\chi u}(\tau)| < +\infty\,. \tag{1.17}$$

One has

Corollary 1.4. *The point $(t_0,\tau_0)\in T^*\mathbb{R}^n-\{0\}$ is not in* WF(u) *if and only if there exists a neighborhood W of $t_0-i\tau_0$ in $\mathbb{C}^n$ such that for every $N\in\mathbb{N}$:*

$$\sup_{x\in W,\ \lambda\ge 1}\lambda^N|Tu(x,\lambda)|e^{-\frac{\lambda}{2}(\mathrm{Im}\,x)^2} < +\infty\,. \tag{1.18}$$

Proof. The condition $(t_0,\tau_0)\notin\mathrm{WF}(u)$ is equivalent to the following assertion: There exists a conic neighborhood Γ of τ_0 in $\mathbb{R}^n-\{0\}$ and a neighborhood U of t_0 in $\mathbb{R}^n$ such that for every $s\in\mathbb{R}$ and every $(t,\tau)\in U\times\Gamma$, $(t,\tau)\notin\mathrm{WF}_s(u)$. On the other hand, condition (1.18) is equivalent to the existence of a neighborhood W of $t_0-i\tau_0$ such that for every $s\in\mathbb{R}$

$$\int_1^{+\infty}\lambda^{\frac{3n}{2}+2s-1}\int_W |Tu(x,\lambda)|^2e^{-\lambda(\mathrm{Im}\,x)^2}\,dL(x)\,d\lambda < +\infty\,. \tag{1.19}$$

The result follows then from Theorem 1.2: one has just to remark that, by inspection of its proof, one may choose in (1.19) a same neighborhood W for every $s\in\mathbb{R}$ as soon as one may take in (1.13) a same cone Γ for every s (and conversely).

The transformation $u\to Tu(x,\lambda)$ may also be used to characterize the analytic wave front set (sometimes called analytic singular spectrum or microsupport) and the Gevrey wave front set of a distribution u (in fact, it had been introduced for the first purpose in [Sj]). Since we will just use this characterization, we choose to take it as a definition here. Its equivalence with the other possible definitions (using inequalities similar to (1.17) or through boundary values of holomorphic functions, or through cohomological tools) may be found – in the case of the analytic singular spectrum – in [Sj], as well as in the work of Bony [Bo0] proving that there is at most one "reasonable" notion of singular spectrum.

Definition 1.5. i) One says that *the point* $(t_0, \tau_0) \in T^*\mathbb{R}^n - \{0\}$ *is not in the analytic wave front set* (or *singular spectrum*) *of* u, SS(u), if there exists a neighborhood W of $t_0 - i\tau_0$ in $\mathbb{C}^n$ and $\varepsilon > 0$ such that

$$\sup_{W\times[1,+\infty[} e^{-\frac{\lambda}{2}[(\operatorname{Im} x)^2-\varepsilon]}|Tu(x,\lambda)| < +\infty . \tag{1.20}$$

ii) One says that (t_0, τ_0) *is not in the Gevrey-s wave front set of* u, $(s \in]1,+\infty[)$, WF$_{G^s}(u)$, if there exists a neighborhood W of $t_0 - i\tau_0$ in $\mathbb{C}^n$ and $\varepsilon > 0$ such that

$$\sup_{W\times[1,+\infty[} e^{-\frac{\lambda}{2}(\operatorname{Im} x)^2+\varepsilon\lambda^{1/s}}|Tu(x,\lambda)| < +\infty . \tag{1.21}$$

We shall conclude this first section by an inversion formula, due to Lebeau, which gives an expression of a distribution u in terms of its FBI transform Tu. We follow Hörmander [H].

Theorem 1.6. *Let u be a compactly supported distribution on $\mathbb{R}^n$. For every $t \in \mathbb{R}^n$ and $r \in]0,1[$ set*

$$u_r(t) = \tfrac{1}{2}(2\pi)^{-n}\int_0^{+\infty} e^{-\frac{\lambda}{2}}\lambda^{n-1}\,d\lambda \int_{|\omega|=1}\Big(1-\langle\omega,\frac{D}{\lambda}\rangle\Big)Tu(t+ir\omega,\lambda)\,d\omega \tag{1.22}$$

where $D = (D_1,\dots,D_n)$ and $D_j = \frac{1}{i}\frac{\partial}{\partial x_j}$.

Then, for every $r \in]0,1]$, u_r is a real analytic function of t, which converges in the sense of distributions towards u when r goes to $1-$.

Proof. The analyticity of u_r follows from (1.2) and from the similar estimate for $|\frac{D_j}{\lambda}Tu(x,\lambda)|$ (which is obtained applying Cauchy's formula on a polydisk with center at x, with radius of order $\frac{1}{\lambda}$).

Let $\phi \in C_0^\infty(\mathbb{R}^n)$. By definition of Tu,

$$\int \phi(t)\Big(1-\langle\omega,\frac{D}{\lambda}\rangle\Big)Tu(t+ir\omega,\lambda)\,dt = \langle u, \Big(1+\langle\omega,\frac{D}{\lambda}\rangle\Big)T\phi(\cdot - ir\omega,\lambda)\rangle \tag{1.23}$$

for every $r \in]0,1[$, the bracket in the right hand side standing for the duality between distributions and C^∞ functions. Let

$$\phi_r(t) = \tfrac{1}{2}(2\pi)^{-n}\int_0^{+\infty} e^{-\frac{\lambda}{2}}\lambda^{n-1}\,d\lambda \int_{|\omega|=1}\Big(1+\langle\omega,\frac{D}{\lambda}\rangle\Big)T\phi(t-ir\omega,\lambda)\,d\omega . \tag{1.24}$$

Since $T\phi(t-ir\omega,\lambda)$ is rapidly decreasing in λ, uniformly with respect to t staying in a compact subset, $\omega \in S^{n-1}$, $r \in [0,1[$, $\phi_r(t)$ is locally uniformly convergent towards $\phi_1(r)$ when $r \to 1-$ as well as all its derivatives. The theorem then follows from:

Lemma 1.7. *For every function* $\phi \in C_0^\infty(\mathbb{R}^n)$, *one has*

$$\phi(t) = \tfrac{1}{2}(2\pi)^{-n} \int_0^{+\infty} \lambda^{n-1}\, d\lambda \int_{|\omega|=1} e^{-\frac{\lambda}{2}} \Big(1 + \langle \omega, \frac{D}{\lambda}\rangle\Big) T\phi(t - i\omega, \lambda)\, d\omega \ . \tag{1.25}$$

Proof. From Fourier inversion formula, we see

$$\phi(0) = \lim_{\varepsilon \to 0+} \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n \times \mathbb{R}^n} e^{is\tau - \varepsilon|\tau|} \phi(s)\, ds\, d\tau \ . \tag{1.26}$$

We will deform the integration contour with respect to τ in the complex domain. For $\sigma \in \mathbb{C}^n$ staying in $|\operatorname{Im}\sigma| < |\operatorname{Re}\sigma|$, one has $\operatorname{Re}\sigma^2 > 0$ and so, one can set $|\sigma| = \sqrt{\sigma^2}$ where we choose the determination of the square root which is positive on the positive half-axis. Take $a > 0$ small enough so that $a|s| < 1$ for every $s \in \operatorname{Supp}(\phi)$. If we put $\sigma = \tau + ias|\tau|$, one has $\operatorname{Re}|\sigma| \geq \text{cst}\,|\tau|$. Since $d\sigma_1 \wedge \cdots \wedge d\sigma_n = (1 + ia\langle s, \frac{\tau}{|\tau|}\rangle)\, d\tau_1 \wedge \cdots \wedge d\tau_n$, Stokes formula applied to (1.26) allows one to replace the real integration contour in τ by $\sigma = \tau + ias|\tau|$, i.e.

$$\phi(0) = \lim_{\varepsilon \to 0} (2\pi)^{-n} \int_{\mathbb{R}^n \times \mathbb{R}^n} e^{is\tau - as^2|\tau| - \varepsilon|\sigma|} \Big(1 + ia\langle s, \frac{\tau}{|\tau|}\rangle\Big) \phi(s)\, ds\, d\tau \ . \tag{1.27}$$

There is a constant $c > 0$, independent of ε, such that

$$\Big|\frac{\partial}{\partial s}(is\tau - as^2|\tau| - \varepsilon|\sigma|)\Big| \geq c|\tau| \ .$$

By integrations by parts, and since ϕ is C^∞, we see that in (1.27) the integral with respect to ds is rapidly decreasing in $|\tau|$, uniformly in ε. Passing to the limit when $\varepsilon \to 0+$ we obtain

$$\phi(0) = (2\pi)^{-n} \int_{\mathbb{R}^n \times \mathbb{R}^n} e^{is\tau - as^2|\tau|} \Big(1 + ia\langle s, \frac{\tau}{|\tau|}\rangle\Big) \phi(s)\, ds\, d\tau \ . \tag{1.28}$$

This identity holds for every small enough real positive number a. But since the right hand side of (1.28) is an holomorphic function of a in the half-plane $\operatorname{Re} a > 0$, (1.28) is true for every such a.

Take $a = \frac{1}{2}$, apply (1.28) to $\phi(t + \cdot)$ and make the change of variable $\tau = -\lambda\omega$, $\lambda \in \mathbb{R}_+^*$, $\omega \in S^{n-1}$. One gets

(1.29) $\phi(t)$

$$= (2\pi)^{-n} \int_0^{+\infty} \lambda^{n-1}\, d\lambda \int_{|\omega|=1} d\omega \int e^{i\lambda\langle t-s,\omega\rangle - \frac{(t-s)^2}{2}} \Big(1 + \frac{i}{2}\langle t - s, \omega\rangle\Big)\, \phi(s)\, ds$$

and equality (1.25) follows.