Lecture Notes in Mathematics
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Multiplication of Distributions

A tool in mathematics, numerical engineering and theoretical physics

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Introduction

The aim of this book is to present a recent mathematical tool, in a way which is very accessible and free from mathematical techniques. The presentation developed here is in part heuristic, with emphasis on algebraic calculations and numerical recipes that can be easily used for numerical solutions of systems of equations modelling elasticity, elastoplasticity, hydrodynamics, acoustic diffusion, multifluid flows. This mathematical tool has also theoretical consequences such as convergence proofs for numerical schemes, existence - uniqueness theorems for solutions of systems of partial differential equations, unification of various methods for defining multiplications of distributions. These topics are not developed in this book since this would have made it not so elementary. A glimpse on these topics is given in two recent research expository papers : Colombeau [14] in Bull. of A.M.S. and Egorov [1] in Russian Math. Surveys. A detailed and careful self contained exposition on these mathematical applications can be found in Oberguggenberger's recent book [11] " Multiplication of distributions and applications to partial differential equations". A set of references is given concerning both the applied and the theoretical viewpoints. This book is the text of a course in numerical modelling given by the author to graduate students at the Ecole Normale Supérieure de Lyon in the academic years 1989 - 90 and 1990 - 91.

Many basic equations of physics contain, in more or less obvious or hidden ways, products looking like "ambiguous multiplications of distributions" such as products of a discontinuous function f and a Dirac mass centered on a point of discontinuity of f or powers of a Dirac mass. These products do not make sense within classical mathematics (i. e. distribution theory) and usually appear as "ambiguous" when considered from a heuristic or physical viewpoint. The idea developed here is that these statements of equations of physics are basically sound, and that a new mathematical theory of generalized functions is needed to explain and master them. Such a theory was first developed in pure mathematics and then it was used in applications ; the mathematician reader can look at the books Colombeau [2, 3], Part II of Rosinger [1], Biagioni [1] and Oberguggenberger [11].

The ambiguity appearing in equations of physics when these equations involve "heuristic multiplications of distributions" corresponds in our theory to the fact that, when formulated in the weakest way, these equations have an infinite number of possible solutions. This recognition of infinitely many solutions was essentially known and understood without our theory (at least in Quantum Field Theory). To resolve the ambiguities our new setting can suggest more precise formulations of the equations (these more precise formulations do not make sense within distribution theory). On physical ground one chooses one of these more precise formulations in which there is no more ambiguity. This technique is developed in this book on various examples from physics. This gives directly new algebraic formulas and new numerical schemes. When one has algebraic jump formulas (for systems in nonconservation form) then it is an easy further step to transfer this knowledge into numerical schemes of the Godunov type. This last numerical technique - Godunov schemes for systems in nonconservation form (elastoplasticity, multifluid flows) or for nonconservative versions of systems of conservation laws (hydrodynamics) - is the main application developed in this text (chapters 4 and 5).

The book is divided into four parts. Part I (chapters 1 and 2) deals with preliminaries from mathematics and physics. Part II (chapter 3) is a smooth introduction to our theory of generalized functions. Part III (chapters 4, 5, 6) is the main part : there new numerical methods are developed ; for simplicity most of them are presented on one dimensional models, but they extend to the 2 and 3 dimensional problems of industrial use or physical significance ; numerical results are presented and references are given. Part IV is made of various complements.

Now let us describe briefly the contents of each chapter. In chapter 1 we introduce our viewpoint, distribution theory and its limitations, in a way convenient for a reader only aware of the concepts of partial derivatives (of functions of several real variables) and of integrals (of continuous functions). Chapter 2 exposes the main equations of Continuum Mechanics considered in the book (hydrodynamics, elastoplasticity, multifluid flows, linear acoustics). The aim of chapter 3 is to describe this new mathematical tool without giving the precise mathematical definitions : the viewpoint there is that these generalized functions can be manipulated correctly provided one has an intuitive understanding of them and provided one is familiar with their rules of calculation. Chapter 4 deals with the classical (conservative) system of fluid dynamics. No products of distributions appear in it , even in case of shock waves. But, surprisingly, our tool gives new methods for its numerical solution : one transforms it into a simpler, but in nonconservative form, system and then one computes a solution from nonconservative Godunov type schemes. In this case, since the correct solution is known with arbitrary precision it is easy to evaluate the value of the new method (by comparison with the exact solution and with numerical results from classical conservative numerical methods). Chapters 5 and 6 deal with systems containing multiplications of distributions that arise directly from physics : nonlinear systems of elastoplasticity and multifluid flows in chapter 5 and linear systems of acoustics in chapter 6. In chapter 7 we expose in the case of a simple model (a self interacting boson field) the basic heuristic calculations of Quantum Field Theory. This topic has been chosen since Quantum Field Theory is the most famous historic example in which the importance of multiplications of distributions was first recognized. Chapter 8 contains a mathematical introduction to these generalized functions and mathematical definitions.

I am particularly indebted to A. Y. Le Roux and B. Poirée. I was working on the multiplication of distributions from a viewpoint of pure mathematics when we met . Their research work (numerical analysis and engineering, physics) had shown them the need of a multiplication of distributions. They introduced me kindly and smoothly to their problems. This was the origin of the present book. I am also very much indebted to L. Arnaud, F. Berger, H. A. Biagioni, L.S. Chadli, P. De Luca, J. Laurens, A. Noussaïr, M. Oberguggenberger, B. Perrot, I. Zalzali for help in works used in the preparation of this book. The main part of the typing has been done by B. Mauduit to whom I also extend my warmest thanks.

Chapter 1. Introduction to generalized functions and distributions

§1.1 THE VIEWPOINT OF THIS BOOK.

Long ago physicists and engineers have introduced formal calculations that work well (Heaviside [1] , Dirac [1]) ; in particular they have introduced the Dirac delta function on $\mathbb{R}$

$$\delta(x) = 0 \quad \text{if} \quad x \neq 0$$

$$\delta(0) = +\infty \text{ (so "large" that } \int_{-\infty}^{+\infty} \delta(x)\, dx = 1).$$

Intuitively the Dirac delta function can be considered as some kind of limit - in a sense to be made precise-of the functions δ^ε when $\varepsilon \to 0$: support of $\delta^\varepsilon \subset [-\eta(\varepsilon), \eta(\varepsilon)]$ with $\eta(\varepsilon) \to 0$ as $\varepsilon \to 0$, $\int \delta^\varepsilon(x)\, dx = 1$).

The theory of distributions (Sobolev [1], Schwartz [1]) has given a rigorous mathematical sense to δ and other objects. But, in some important cases, in which the calculations of physicists are complicated (and give results in agreement with those from experiments at the price of ad hoc manipulations : for instance renormalization theory in Quantum Field Theory), the theory of distributions fails. One observes that these formal calculations involve unjustified products of distributions, such as δ^2, the square of the Dirac delta function. L. Schwartz [2] has proved in 1954 the "impossibility of the multiplication of distributions", even in a framework rather disjoint from the theory of distributions. He has proved the non-existence of a differential algebra A (of any kind of "generalized functions" on $\mathbb{R}$) containing the algebra $\mathcal{C}(\mathbb{R})$ (of continuous functions on $\mathbb{R}$) as a subalgebra, preserving the differentiation of functions of class $\mathcal{C}^1$ (i. e. the differentiation in A coincides with the classical one) and having a few other natural properties (Leibniz's rule for the differentiation of a product, the constant function 1 is the neutral element in A for the multiplication, A contains some version of the Dirac delta function). Thus the theory of distributions is not really concerned with this impossiblility result and it appears that the roots of the impossibility go back as far as some incoherence between the multiplication and the differentiation in the setting of $\mathcal{C}^1$ and continuous functions. Since the emergence and recent development of computer science other kinds of "multiplications of distributions " are successfully treated numerically, to model various problems from Continuum Mechanics (elasticity, elastoplasticity, hydrodynamics, acoustics, electromagnetism, ... see the sequel of this book).Therefore, there is presently a situation of impasse between (rigorous) mathematics from one side, theoretical physics and engineering from the other side.

Even, from a viewpoint internal to mathematics, one also faces a major problem : for most systems of partial differential equations (including those modelling the more usual physical situations) distribution solutions are unknown ; even, in many cases, one can prove - often trivially - the nonexistence of distribution solutions. This motivates the introduction of new mathematical objects (for instance the concept of "measure valued solutions to systems of conservation laws", see Di Perna [1], Di Perna - Majda [1], ...). This need, although internal to mathematics, is indeed closely related to the needs from physics and engineering (described above) since these explicit calculations or these numerical recipes are nothing other than attempts for the resolution of equations.

We shall present a mathematical theory of generalized functions, in which the main calculations and numerical recipes alluded to above make sense. This theory has recently been used in Continuum Mechanics, for problems involving "multiplications of distributions". It permits to understand the nature of the problems involved in these multiplications and it leads to new algebraic formulas (jump conditions for systems in nonconservative form), and new numerical methods. From a more theoretical viewpoint this theory gives solutions for previously unsolvable equations. In physically relevant cases these solutions can be indeed classical discontinuous functions (representing shock waves) which are not solutions within distribution theory. Since this theory is recent its limitations are still unknown and we propose numerous research directions (applied and theoretical).

If Ω denotes any open set in $\mathbb{R}^n$ we shall define a new concept of generalized functions on Ω (real valued or complex valued, even vector valued if needed ; we consider that they are real valued, unless the converse is explicitly stated). The set of these generalized functions is denoted by $\mathcal{G}(\Omega)$; $\mathcal{G}(\Omega)$ is a <u>differential algebra</u> (i. e. it has the same operations and rules as the familiar differential algebra $\mathcal{C}^\infty(\Omega)$ of all $\mathcal{C}^\infty$ functions on Ω). If $\mathcal{D}'(\Omega)$ is the vector space of all distributions on Ω (whose definition will be recalled in the next section) one has the inclusions

$$\mathcal{C}^\infty(\Omega) \subset \mathcal{D}'(\Omega) \subset \mathcal{G}(\Omega).$$

$\mathcal{G}(\Omega)$ induces on $\mathcal{D}'(\Omega)$ its addition, scalar multiplication and differentiation (there is no general multiplication in $\mathcal{D}'(\Omega)$). $\mathcal{G}(\Omega)$ induces on $\mathcal{C}^\infty(\Omega)$ all the operations in $\mathcal{C}^\infty(\Omega)$, in particular the multiplication. Thus these generalized functions look as some "super concept of $\mathcal{C}^\infty$ functions". The product of two arbitrary elements of $\mathcal{D}'(\Omega)$ will be in $\mathcal{G}(\Omega)$, not in $\mathcal{D}'(\Omega)$ in general. The connection with the Schwartz impossibility result is - as this could be expected - rather subtle. The algebra $\mathcal{C}(\Omega)$ of all continuous functions on Ω is not a subalgebra of $\mathcal{G}(\Omega)$: the product in $\mathcal{G}(\Omega)$ of two arbitrary continuous functions on Ω does not always coincide with their classical product. The subtility lies in that the difference (in $\mathcal{G}(\Omega)$) between these two products is - in some sense to be made precise after $\mathcal{G}(\Omega)$ will be defined - "infinitesimal" (although nonzero). Being "infinitesimal" this difference can be considered as null as long as it is not multiplied by some "infinite quantity" (infinite quantities like the value $\delta(0)$ of the Dirac delta function at the origin make sense in our setting). In all classical calculations dealing with continuous functions there do not appear such "infinite quantities" and so the difference between the two products of continuous functions is then always insignificant. The new theory is totally coherent with classical analysis and, at the same time, it escapes from Schwartz impossibility result.

The above should not sound too much mysterious, since physicists and mathematicians are indeed familiar with certain aspects of this subtility. Let us consider the following classical remark from shock wave solutions of systems of conservation laws (Richtmyer [1]).Consider the equation

(1) $$u_t + uu_x = 0$$

and seek a travelling wave solution : i. e. $u(x, t) = a$ for $x < ct$, $u(x, t) = a+b$ for $x > ct$, a, b, constants, c is the constant velocity of the shock. Such a solution can be written as

(2) $$u(x, t) = b\,Y(x - ct) + a$$

where Y is the Heaviside step function ($Y(\xi) = 0$ if $\xi < 0$, $Y(\xi) = 1$ if $\xi > 0$). Interpreting (1) as

$$u_t + \frac{1}{2}(u^2)_x = 0$$

(2) gives (Y' is the derivative of Y in the sense of distributions, see next section)

$$-bc\,Y'(x - ct) + \frac{1}{2}(b^2\,Y(x - ct) + 2ab\,Y(x - ct) + a^2)_x = 0$$

since one has $Y^2 = Y$ (in the algebra of piecewise constant functions). One obtains (since Y' is non zero)

(3) $$c = a + \frac{b}{2}.$$

Now let us multiply (1) by u : this gives

(4) $$uu_t + u^2u_x = 0$$

that can be naturally interpreted as

(4') $$\frac{1}{2}(u^2)_t + \frac{1}{3}(u^3)_x = 0.$$

Putting (2) into (4) one gets for c a value different from (3). One concludes that (1) and (4) have different shock wave solutions : thus the correct statement of the equations has to be carefully selected on physical ground, see Richtmyer [1]. The passage from (1) to (4) is a multiplication, and so we have put in evidence some incoherence between multiplication and differentiation. This incoherence is reproduced in the following calculations. Classically one has

(5) $Y^n = Y \qquad \forall n = 2, 3, \ldots$

Differentiation of (5) gives

(6) $n\, Y^{n-1}\, Y' = Y'$

thus one has

(7) $2Y\, Y' = Y'$.

Multiplication by Y gives

$$2Y^2Y' = YY'.$$

Using (6) one gets

$$\frac{2}{3}\, Y' = \frac{1}{2}\, Y'$$

which is absurd since $Y' \neq 0$. Of course the trouble arises at the origin, since this is the unique singular point of Y and Y'. If one accepts to consider $Y^n \neq Y$ (n = 2, 3, ...) there is no more trouble. Of course Y^n - Y is "infinitesimal" - in a sense to be made precise later, so that if- instead of multiplying it by Y' - one multiplies it by some more reasonable function then one gets still an "infinitesimal" result ; in this latter case one could as well have considered $Y^n = Y$, as classically. This examplifies the general fact that the theory of generalized functions in $\mathcal{G}(\Omega)$ can be considered as a refinement of classical analysis , without any contradiction with classical analysis and distribution theory (as long as one considers only calculations valid inside distribution theory) ; the above example of multiplication by u in (1) does not make sense inside distribution theory. These calculations would a priori make sense in $\mathcal{G}$ but $Y^n \neq Y$ in $\mathcal{G}(\mathbb{R})$ as soon as $n \neq 1$. In view of that it appears that the assumption that the classical algebra $\mathcal{C}_f(\mathbb{R})$ (of all piecewise constant functions) is a subalgebra of A (underlying in Schwartz's impossibility result, see § 1. 3) can be considered as irrealistic.

Research problem. The reader is assumed to know the definition of $\mathcal{G}(\Omega)$ and Nonstandard Analysis. Clarify the connections between our concept of generalized functions and the nonstandard functions. Since there is no canonical inclusion of $\mathfrak{D}'(\Omega)$ into the set of nonstandard functions Nonstandard Analysis is probably much closer to the simplified concept $\mathcal{G}_s(\Omega)$ defined below in § 8.4. Various constructions of Nonstandard Analysis mimicking the construction of $\mathcal{G}(\Omega)$ are given in Oberguggenberger [9] and Todorov [1]. Since both theories realize a differential and integral calculus dealing with infinitesimal and infinitely large quantities it seems to me that a fusion (of both theories) is perhaps possible.For a comparison of the two theories in the context of nonlinear hyperbolic equations see Oberguggenberger [12].

§1. 2 AN INTRODUCTION TO DISTRIBUTIONS. This section is intended to the reader who does not know distribution theory ; it can be dropped by the other readers. If Ω is an open set in $\mathbb{R}^n$ we denote by $\mathfrak{D}(\Omega)$ the vector space of all (scalar valued) $\mathcal{C}^\infty$ functions on Ω with compact support (such functions exist ! ; the support of a function f (denoted by supp f) is the closure of $\{x \mid f(x) \neq 0\}$). We say that a sequence (f_n) of functions in $\mathfrak{D}(\Omega)$ "tends to 0" (notation "$f_n \to 0$") if and only if 1) and 2) below are satisfied :

1) their supports are contained in a fixed compact subset of Ω
2) for every partial derivative D (including the identity)

$$\lim_{n\to\infty} \sup_{x\in\Omega} |D f_n(x)| = 0.$$

Definition. A distribution on Ω is a linear form $T : \mathfrak{D}(\Omega) \to \mathbb{C}$ such that $T(f_n) \to 0$ in $\mathbb{C}$ as soon as "$f_n \to 0$".

We denote by $\mathfrak{D}'(\Omega)$ the space of all distributions on Ω. $\mathfrak{D}'(\Omega)$ is a vector space.

We define partial derivatives of distributions by : if $T \in \mathfrak{D}'(\Omega)$, $\frac{\partial T}{\partial x_i} \in \mathfrak{D}'(\Omega)$ is the distribution defined by $\frac{\partial T}{\partial x_i}(\varphi) = -T\left(\frac{\partial \varphi}{\partial x_i}\right) \quad \forall \varphi \in \mathfrak{D}(\Omega)$;

thus

$$DT(\varphi) = (-1)^{o(D)}\, T(D\varphi) \quad \forall \varphi \in \mathfrak{D}(\Omega)$$

if D is an arbitrary partial derivative operator and if o(D) is its order.

We multiply a $\mathcal{C}^\infty$ function and a distribution according to the formula : if $\alpha \in \mathcal{C}^\infty(\Omega)$ and $T \in \mathfrak{D}'(\Omega)$, the product $\alpha . T \in \mathfrak{D}'(\Omega)$ is defined by

$$(\alpha . T)(\varphi) = T(\alpha\varphi) \quad \forall \varphi \in \mathfrak{D}(\Omega).$$

Any locally integrable function is a distribution : if $f \in L^1_{loc}(\Omega)$ then we set

$$f(\varphi) = \int_\Omega f(x)\, \varphi(x)\, dx \qquad \forall\, \varphi \in \mathcal{D}(\Omega).$$

Since it is known that $f(\varphi) = 0 \;\; \forall\, \varphi \in \mathcal{D}(\Omega) \Rightarrow f = 0$ in $L^1_{loc}(\Omega)$ one has an inclusion $L^1_{loc}(\Omega) \subset \mathcal{D}'(\Omega)$. If $p = 2, 3, \ldots, \infty$ one has similarly an inclusion $L^p_{loc}(\Omega) \subset \mathcal{D}'(\Omega)$.

One checks at once that the differentiation of a distribution and the multiplication by a $\mathcal{C}^\infty$ function extend these respective classical operations (in $\mathcal{C}^1(\Omega)$ and in $L^p_{loc}(\Omega)$ respectively). However note that if f is a classical function which is twice differentiable (but not twice continuously differentiable) and such that $f''_{x,y} \neq f''_{y,x}$ (such functions exist !) then since $f''_{x,y} = f''_{y,x}$ in the sense of distributions, the classical and distributional second derivatives are not identical.

Example 1 : the Dirac delta distribution defined by the formula $\delta(\varphi) = \varphi(0)$. If $\delta^\varepsilon \in \mathcal{C}(\mathbb{R})$,

$0 < \varepsilon < 1$, $\delta^\varepsilon \geq 0$, $\int \delta^\varepsilon(x)\, dx = 1$, supp $\delta_\varepsilon \subset [-\varepsilon, +\varepsilon]$, then if $\varphi \in \mathcal{D}(\mathbb{R})$

$$\int_{\mathbb{R}} \delta^\varepsilon(x)\, \varphi(x)\, dx = \int_{\mathbb{R}} \delta^\varepsilon(x)\, (\varphi(0) + x\varphi'(\theta x))\, dx \to \varphi(0) \text{ as } \varepsilon \to 0 \;\; (0 < \theta < 1).$$

One says that $\delta^\varepsilon \to \delta$ in $\mathcal{D}'(\mathbb{R})$ as $\varepsilon \to 0$.

Example 2. the derivative of the Heaviside function : prove that the derivative Y' of the Heaviside function Y is the Dirac distribution δ.

One can prove (Schwartz [1])the structure theorem :

Theorem . Any distribution is locally a partial derivative of a continuous function.

In other words : $\forall\, T \in \mathcal{D}'(\Omega)\; \forall x_0 \in \Omega\; \exists$ an open neighborhood V_{x_0} of x_0 in Ω, $\exists\, f \in \mathcal{C}(V_{x_0})$ and $\exists$ a partial derivation operator D such that

$$T_{|V_{x_0}} = Df \quad \text{in } \mathcal{D}'(V_{x_0})$$

where $T_{|V_{x_0}}$ is the restriction of T to V_{x_0} (obvious definition : one considers only the test functions $\varphi \in \mathcal{D}(V_{x_0}) \subset \mathcal{D}(\Omega)$).

From this structure theorem the distributions constitute the smallest space in which it is permitted to differentiate (infinitely) all continuous functions (and also all L^p_{loc} functions p = 1, 2, ..., ∞.).

Finally the distributions enjoy essentially all the nice properties of the $\mathcal{C}^\infty$ functions, with the basic exceptions of the multiplication (as well as all main nonlinear operations ; try to multiply "reasonably" Y and δ, δ and δ,..), of the restriction to a vector space (let δ_2 be the Dirac distribution on $\mathbb{R}^2$: $\delta_2(\varphi) = \varphi(0,0)$; try to restrict δ_2 to $\mathbb{R} \times \{0\}$), and of the composition product (try to define the composition $f_o\delta$ ($f \in \mathcal{C}^\infty(\mathbb{R})$).

Various extensions of the distributions have been proposed.

The ultradistributions. They are defined by replacing $\mathcal{D}(\Omega)$ by a smaller space of $\mathcal{C}^\infty$ functions, satisfying for instance, in one dimension, bounds of the kind

$$\| \varphi^{(k)} \|_\infty \leq M C^k (k!)^s, s > 1$$

(for s = 1 the function φ is analytic and so cannot have compact support unless it is the zero function). Various spaces of ultra-distributions are defined as the duals of such subspaces of $\mathcal{D}(\Omega)$; these spaces contain $\mathcal{D}'(\Omega)$ but do not have very different properties ; see Lions-Magenes [1] for definitions and references.

The analytic functionals. One considers a space of analytic functions, for instance the space $\mathcal{H}(\Omega)$ of all holomorphic functions on an open set $\Omega \subset \mathbb{C}^n$, equipped with the topology of uniform convergence on the compact subsets of Ω. The space of analytic functionals is defined as the dual $\mathcal{H}'(\Omega)$. Since any analytic function with compact support is the constant 0 there are difficulties to define the support of an analytic functional ; further one can only multiply the analytic functionals by analytic functions. See Martineau [1].

The hyperfunctions generalize both the distributions and the analytic functionals, see Martineau [2]. Grosso modo a hyperfunction on $\mathbb{R}^n$ appears as a locally finite series of analytic functionals that patch together.

There are linear PDEs without distribution solutions, but that have solutions which are analytic functionals or hyperfunctions. However all these extensions of the distributions share essentially the same properties : unlimited differentiation but impossibility of the multiplication in general ; also, like in the setting of distributions, many very simple linear PDEs with polynomial coefficients do not have solutions in these spaces, see § 1. 4 below.

Research Problem. The reader is assumed to know the definition of $\mathcal{G}(\Omega)$; it is clear that this definition can be modified so as to include the ultra-distributions (and so to permit a general multiplication of ultra-distributions). Is it possible - probably at the price of a greater modification in the definition of $\mathcal{G}(\Omega)$ - to include the analytic functionals and / or the hyperfunctions in a differential algebra looking like $\mathcal{G}(\Omega)$ (thus permitting a general multiplication of analytic functionals and / or hyperfunctions)? A special type of ultradistributions has been included in a larger algebra in Gramchev [1].

§1. 3 SCHWARTZ IMPOSSIBILITY RESULT.

Theorem [Schwartz [2], 1954]. Let A be an algebra containing the algebra $\mathcal{C}(\mathbb{R})$ of all continuous functions on $\mathbb{R}$ as a subalgebra. Let us assume that the constant function $1 \in \mathcal{C}(\mathbb{R})$ is the unit element in A. Further let us assume that there exists a linear map $D : A \rightarrow A$ generalizing the derivation of continuously differentiable functions and satisfying Leibniz's rule ($D(a.\,b) = Da.\,b + a.\,Db$). Then one has $D^2(|x|) = 0$.

Of course D (|x|) has values -1 for $x < 0$ and +1 for $x > 0$, therefore $D^2(|x|)$ should be null outside 0, "infinite" in 0 so that

$\int_{-\infty}^{+\infty} D^2(|x|)\,dx = [D(|x|)]_{-\infty}^{+\infty} = 2$. Thus the conclusion of the theorem contradicts any reasonable intuition. In distribution theory $D^2(|x|) = 2\delta$ and so the above result shows that A cannot contain the Dirac delta function, thus making the algebra A uninteresting.

Basic Remark. The distributions are not involved in Schwartz's impossibility result : the algebra $\mathcal{C}(\mathbb{R})$, the differentiation of continuously differentiable functions, and the usual calculation rules are the only ingredients that produce the impossibility. And all these ingredients are perfectly natural ! However it has already been noticed in §1.1 that the multiplication of piecewise constant functions together with the usual rules of differentiation produces at once a contradiction.

Before the proof we give a lemma.

Lemma In A $\quad xa = 0 \Rightarrow a = 0$ (where x is the classical function $x \rightarrow x$ and where a is an arbitrary element of A).

Proof of the theorem.

$$D(x|x|) = Dx\,.\,|x| + x.\,D(|x|) = |x| + x.D(|x|).$$

Therefore

$$D^2(x|x|) = 2D(|x|) + xD^2(|x|).$$

In $\mathscr{C}^1(\mathbb{R})$, hence in A :

$$D(x|x|) = 2\,|x|.$$

Therefore

$$D^2(x|x|) = 2D|x|.$$

It follows from the two above expressions for $D^2(x|x|)$ that x. $D^2(|x|) = 0$ thus from the lemma $D^2(|x|) = 0$

□

Proof of the lemma. The functions $x(\log|x| - 1)$ and $x^2(\log|x| - 1)$ are in $\mathscr{C}(\mathbb{R})$ provided we give them the value 0 for $x = 0$. Using Leibniz's rule in A

$D\{x(\log|x|-1).x\} = D\{x(\log|x|-1)\}.x + x(\log|x|-1)$
$D^2\{x(\log|x|-1).x\} = D^2\{x(\log|x|-1)\}.x + 2\,D\{x(\log|x|-1\}.$
Thus
(8) $\quad D^2\{x(\log|x|-1)\}.x = D^2\{x^2(\log|x|-1)\} - 2D\{x(\log|x|-1)\}.$

But, since D coincides with the usual derivation operator on $\mathscr{C}^1$ functions and since the function $x^2(\log|x| - 1)$ is a $\mathscr{C}^1$ function :

$D\{x^2(\log|x|-1)\} = 2x(\log|x|-1) + x.$

Therefore in A
(9) $\quad D^2\{x^2(\log|x|-1)\} = 2D\{x(\log|x|-1)\} + 1.$

(8) and (9) yield :

$$D^2\{x(\log|x|-1)\}.x = 1.$$

To simplify the notation set $y = D^2\{x(\log|x|-1\}$; then $y.x = 1$; thus $x.a = 0 \Rightarrow y\,(xa) = 0 \Rightarrow (yx)a = 0 \Rightarrow 1.a = 0 \Rightarrow a = 0$.

□

A more detailed discussion is given in Rosinger [1] Part I chap 2.

Research problem. Many particular multiplications of distributions have been considered, see Colombeau [1] chap 2, Rosinger [1] App. 5 in Part 2, Oberguggenberger [11]. Up to now it has been proved that nearly all of them are particular cases (modulo some concept of "infinitesimality" as for the product of continuous functions, see chapter 8) of the multiplication in $\mathcal{G}(\Omega)$, see Oberguggenberger [1], Jelinek [1,2]. There remains some possible studies in this field.

§1.4 LINEAR PDEs WITHOUT DISTRIBUTION SOLUTION.

It is immediate to show that certain Cauchy problems do not have solution : for instance the equation

$$\begin{cases}(\frac{\partial}{\partial t} + i\frac{\partial}{\partial x})u = 0\\ u(x,0) = u_0(x)\end{cases}$$

cannot have C^1(and also any distribution) solution in an open set Ω intersecting the line $t = 0$ if u_0 is $\mathcal{C}^\infty$ but not analytic. Indeed a solution u would be holomorphic in Ω(one can prove that if $u \in \mathcal{D}'(\Omega)$, $\Omega \subset \mathbb{C}$ open, then $\frac{\partial}{\partial \bar{z}} u = 0 \Rightarrow u \in \mathcal{H}(\Omega)$). Therefore u_0 would be a real analytic function. The above equation cannot also have solutions in $\Omega \cap t > 0$, the initial condition being understood as a limit when $t \to 0$: extend this solution to $t < 0$ by setting $u(x,t) = \overline{u(x,-t)}$ (where the bar denotes complex conjugation) and apply the proof above. But one can prove (Hörmander [1]) that for any $\mathcal{C}^\infty$ function f on $\mathbb{R}^2$ there is $u \in \mathcal{C}^\infty(\mathbb{R}^2)$ such that $(\frac{\partial}{\partial t} + i\frac{\partial}{\partial x})\, u = f$. Considerable effort has been invested on the following problem. Let

$$P(x,D) = \sum_{\substack{p\in \mathbb{N}^n\\ |p|\leq m}} c_p(x)\, D^p, \quad (c_p\in \mathcal{C}^\infty(\mathbb{R}^n),\ D^p=\frac{\partial^{|p|}}{\partial {x_1}^{p_1}\ldots\partial {x_n}^{p_n}})$$

be a nonzero linear partial differential operator with $\mathcal{C}^\infty$ coefficients.

Problem. Let $f \in \mathcal{C}^\infty(\mathbb{R}^n)$ and let $x_0 \in \mathbb{R}^n$ be given ; is there V_{x_0}, open neighborhood of x_0, and $u \in \mathcal{D}'(V_{x_0})$ such that

$$P(x,D)\ u = f \quad \text{in } V_{x_0}?$$

H. Lewy [1] has produced a celebrated counterexample. Let $x_1,x_2,y_1 \in \mathbb{R}$ and let us consider the equation

(L) $$\left[-\frac{\partial}{\partial x_1} - i\frac{\partial}{\partial x_2} + 2i(x_1 + ix_2)\frac{\partial}{\partial y_1}\right] u = f(y_1).$$

Then for any f which is $\mathcal{C}^\infty$ and not analytic, and any point $x_o \in \mathbb{R}^3$ equation (L) provides a negative answer to the above problem.

<u>Sketch of proof :</u> for simplification we shall only consider the case $x_o = (0,0,y_1^0)$ and the case u is a $\mathcal{C}^1$ function (easy extension to the case u is a distribution). Set $x_1 + ix_2 = y_2^{1/2} e^{i\theta}$, $y_2 > 0$ and let

$$U(y_1,y_2) = i\int_0^{2\pi} e^{i\theta} y_2^{1/2} u(x_1,x_2,y_1)\, d\theta.$$

From (L) one computes (Lewy [1]) that

$$\frac{\partial U}{\partial y_1} + i\frac{\partial U}{\partial y_2} = \pi f(y_1).$$

Let F be real such that F' = f ; then the function

$$V(y_1,y_2) = U(y_1,y_2) - \pi F(y_1)$$

is $\mathcal{C}^1$ and satisfies

$$\frac{\partial V}{\partial y_1} + i\frac{\partial V}{\partial y_2} = 0$$

in the intersection of an open ball in the (y_1,y_2) plane centered at $(y_1^0, 0)$ and the half plane $y_2 > 0$; thus it is holomorphic in this upper half ball. Further $U(y_1, 0) = 0$ and $V(y_1,0) = -\pi F(y_1)$ and so V is real valued on $y_2 = 0$. Thus V can be continued holomorphically in the whole of the open ball ; since $F(y_1) = -\frac{1}{\pi} V(y_1,0)$ F is an analytic function and so f is analytic. □

A similar counterexample has been given in the space of hyperfunctions (larger than the space of distributions), see Shapira [1]. Since then a great amount of work has been devoted to the research of necessary and of sufficient conditions for local solvability, see Hörmander [2].

More details on the contents of this section, and other equations without solution, are given in Rosinger [1] Part1 chap.3.

Research problem. It has been proved that also in $\mathcal{G}$ linear PDEs may fail to have solutions. Thus the problem is to find - still in $\mathcal{G}$ or in a similar setting of generalized functions - a convenient formulation of the PDEs, even the linear ones with $\mathcal{C}^\infty$ coefficients, allowing general existence results. Of course coherence with the classical solutions - when they exist - should be obtained. An attempt is presented in Rosinger [1] Part II chap 3, Egorov [1], Colombeau-Heibig-Oberguggenberger [1].

Chapter 2. **Multiplications of distributions in classical physics.**

§2. 1 ELASTICITY AND ELASTOPLASTICITY. In this section we consider large deformations of solid bodies, that could be produced for instance by a strong collision. These large deformations may lead to plastic or other forms of structural failure. At the level of numerical computations this imposes a Eulerian description (i. e. with a fixed frame of reference) since the Lagrangian description (i. e. with a frame of reference following the deformations of the medium) is subject to numerical failure at large deformations. Experience has shown that Eulerian methods can work very well. The system of equations modeling the behaviour of solids includes at first the basic classical laws of conservation of mass, momentum and energy ; usually viscosity is neglected. The basic conservation laws are completed by "constitutive equations" obtained from experiments on each material. The constitutive equations can take very different forms (they distinguish steal from rubber since the conservation laws are the same !). In this text we limit ourselves to the simplest models of elasticity and elastoplasticity, which can be stated as follows (see Arnaud [1] for instance). At first we begin with the purely elastic case.

Notation. $x = (x_1,x_2,x_3)$ = space coordinate, t = time .

ρ = density

$\vec{U} = (u_1,u_2,u_3)$ = velocity vector

$\overline{\Sigma}$ = stress tensor,with components $\sigma_{i,j}, 1 \leq i,j \leq 3$

$p = -\frac{1}{3}\,\mathrm{trace}\,(\overline{\Sigma}) =$ pressure

$\mathbb{1}$ = identity 3 x 3 matrix

$\overline{S} = \overline{\Sigma} + p\,\mathbb{1}$ = stress deviation tensor

$\overline{V}$ = rate of deformation tensor, of components $v_{i,j} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right), 1 \leq i, j \leq 3$

$\overline{\Omega}$ = spin tensor, of components $\omega_{i,j} = -\frac{1}{2}\left(\frac{\partial u_j}{\partial x_i} - \frac{\partial u_i}{\partial x_j}\right)$

e = total specific energy

$I = e - \frac{1}{2}\,\vec{U}.\vec{U}$ = internal specific energy

$\frac{d}{dt}$ denotes the particle time derivative : if f is a function of (x,t) then

$$\frac{d}{dt} f = \frac{\partial f}{\partial t} + \vec{U}.\,\overrightarrow{\mathrm{grad}}\, f$$

if $\overline{A} = (a_{ij})$ is a 3 x 3 tensor we set