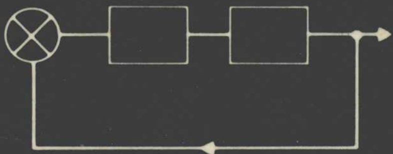

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Mathematical Theories of Nonlinear Systems

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MATHEMATICAL THEORIES OF NONLINEAR SYSTEMS

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MATHEMATICAL THEORIES OF NONLINEAR SYSTEMS

(infinite-dimensional) bilinear systems. Hence we may obtain bilinear approximations to many types of nonlinear systems.

In Chapter 4 we explore the relation between bilinear representations of nonlinear systems and their Volterra series expansions. Moreover, it is shown that bilinear systems have many of the properties of linear systems (or these properties can be generalized to bilinear systems) and so it will be seen that such systems form a very important extension to the class of linear systems. We also present an approach to the frequency-domain theory of nonlinear systems which gives rise to the notion of poles and zeros. A different theory of nonlinear system zeros can be found in Nijmeijer and Schumacher, 1985.

Finally, in Chapter 5, we present an introduction to nonlinear distributed parameter systems. Because of the importance of bilinear systems in finite-dimensional space, we restrict attention mainly to this kind of system. This means that we can work on a (flat) Hilbert or Banach space. One of the main outstanding problems in nonlinear systems theory is to derive results similar to those in Chapters 2–4 for infinite-dimensional systems defined on Banach manifolds.

Since we work throughout in a differentiable category, the methods developed here do not apply to engineering systems containing hysteresis, dead zones, discontinuities, etc. Moreover we do not discuss discrete systems, although much of what we present here holds, when suitably modified, for discrete systems. The third major area of systems theory which is not covered here is that of stochastic systems. Differential-geometric methods used here for the study of deterministic systems can also be applied to stochastic systems; we refer the reader to Marcus, 1984 or Collingwood, 1985.

S.P.B.

LIST OF SYMBOLS

$\mathcal{O}$	set of open sets in a topological space
$\subseteq$	'subset of'
$\in$	'belongs to'
$\ni$	'contains the element'
$B_\varepsilon(x)$	ball of radius ε, centre x
2^X	sets of subsets of X
$X/\sim$	quotient set
$\mathbb{R}^n$	n-dimensional Euclidean space
$\\|\cdot\\|$	norm
T^n	n-dimensional torus
$f \circ g$	composition of functions f, g
$\triangleq$	'equal by definition'
$C^\infty(M)$ or $\mathscr{F}(M)$	set of real-valued functions on M
$f\mid U$	restriction of f to U
$\overline{\mathbb{R}^1}$	$\mathbb{R}^1 \cup \{-\infty, \infty\}$
$\square$	end of theorem, etc.
$GL(n)$	general linear group of degree n
$SL(n)$	special linear group of degree n
$O(n)$	orthogonal group of degree n
$Symm(n)$	set of n-dimensional symmetric matrices
$SO(n)$	special orthogonal group of degree n
$SO(p, q)$	special orthogonal group of type (p, q)
$Sp(n)$	symplectic group of degree n
$T_p(M)$	tangent space to M at P
$(\mathrm{d}f)_p$	differential of f at p
$(\mathrm{T}f)_p$ or f_{*p}	differential of $f: M \to N$ at p
$\otimes$	tensor product
T_p^*	cotangent space at p
f_p^*	dual map of f

$\cong$	'isomorphic to'
$\oplus$	direct sum
T^r_s	tensor space of type (r, s)
$L_r(V; F)$	set of r-linear maps of $V \times \cdots \times V$ into F
T^*	set of contravariant tensors
T_*	set of covariant tensors
$[X, Y]$	Lie bracket of X, Y
$\mathscr{T}^*_{*,p}$	tensor space at p
Ω_*	set of exterior differential forms
A_s	alternation map
$\wedge$	exterior multiplication
(E, p, B)	general bundle
$T(M)$	tangent bundle
$G_k(\mathbb{R}^n)$	Grassmann manifold
$V_k(\mathbb{R}^n)$	frame bundle
$E(\gamma^n_k)$	universal bundle
f^*E	pull-back bundle
$M^k_{k\times n}$	set of $k \times n$ matrices of rank k
$\mathbb{P}_{n-1}(\mathbb{R})$	real projective space
$\mathbb{P}_{n-1}(\mathbb{C})$	complex projective space
$\mathrm{Hom}(E, F)$	homomorphism bundle
Φ	flow of a vector field
L_X	Lie derivative with respect to x
$\lrcorner$	interior multiplication
$\mathscr{L}$, $\mathfrak{M}$, $\mathfrak{N}$	distributions
$\mathscr{T}^k_x$	subset of T_x
L_a, R_a	left and right translations
$\mathscr{I}_a$	inner automorphism of G
$\mathfrak{g}, \mathfrak{h}, \mathfrak{a}, \mathfrak{b}$	Lie algebras
ad, Ad	adjoint maps
$\mathfrak{D}\mathfrak{g}$	derived algebra of $\mathfrak{g}$
$\mathscr{C}^k\mathfrak{g}$	central series of $\mathfrak{g}$
$\mathfrak{g}^\alpha$	root space of $\mathfrak{g}$
Δ	nonzero roots
$\mathbb{C}^n$	n-dimensional complex space
$\mathscr{D}_Y(p)$	discriminant variety of p with respect to Y
$\sqrt{I}$	radical of I
(p)	principal ideal
$\bar{I}$	closure of I
$\cup, \cap, \bigcup, \bigcap$	union, intersection
$\langle .\, , . \rangle$	inner product
X^*	dual space of X
$\langle .,. \rangle_{H^*,H}$	duality in H^*, H

$C^k(\bar{\Omega})$	functions k times differentiable on $\bar{\Omega}$
$\|\cdot\|_k$	norm on previous space
$L^p(\Omega)$	functions with integrable p^{th} power
$\|\cdot\|_{L^p(\Omega)}$	norm on previous space
$\|\cdot\|_{L_\infty(\Omega)}$	norm on functions essentially bounded on Ω
$\langle .,. \rangle_{L^2(\Omega)}$	inner product on $L^2(\Omega)$
ℓ^p	pth power summable sequences
$\mathscr{D}'(\Omega)$	space of distributions on Ω
$H^{p,m}(\Omega)$	Sobolev space
$\|\cdot\|_{p,m}$	norm on previous space
$H_0^{p,m}(\mathbb{R}^n)$	Sobolev space
$\mathscr{S}'(\mathbb{R}^n)$	space of tempered distributions
$M^\perp$	orthogonal complement of m
$\mathscr{B}(X, Y)$	bounded operators from X to Y
$\rho(A)$	resolvent set of A
$\sigma_P(A)$	point spectrum of A
$\sigma_C(A)$	continuous spectrum of A
$\sigma_R(A)$	residual spectrum of A
$\mathscr{U}$	input space
$\mathscr{Y}$	output space
$\mathfrak{M}$	space of solution trajectories
$\mathscr{L}(y)$	Laplace transform of y
$\mathscr{L}(V)$	Lie algebra generated by V
$\mathscr{L}_0(V)$	subspace of $\mathscr{L}(V)$
$\mathscr{L}'(V)$	subspace of $\mathscr{L}(V)$
$\mathscr{L}(V)$	subspace of $\mathscr{L}(V)$
$\mathbf{i} = (i_1, \ldots, i_m)$	m-tuple of integers
$\mathbf{t} = (t_1, \ldots, t_m)$	m-tuple of reals
$I_0(V, x)$	integral manifold of $\mathscr{L}_0(V)$
$I_0^t(V, x)$	integral manifold of $\mathscr{L}_0(V)$
$\otimes_k$	k^{th} order tensor product
$\mathscr{P}, \mathscr{I}(\mathscr{P})$	subalgebras of $\mathscr{L}(V)$
$\mathscr{R}(X; \mathscr{B}_0)$	X-radical of $\mathscr{B}_0$
$\Delta^\perp$	codistribution
$\mathscr{X}_{X_1}, \mathscr{Y}_{X_1}, \mathscr{T}_{X_1}, \mathscr{G}_{X_1}$	function spaces
$\ominus_V$	cascade operator
Φ^ξ, ξ_M^Φ	induced actions on M
Γ_ω	vector bundle mappings
$\#, \flat$	1-form to vector field translations
$\mathbb{N}^n$	n-tuples of natural numbers
ℓ_e^1	extended ℓ^1 space
$\mathscr{L}_n$	space of rank-n tensors
$\mathscr{L}_n^T$	space of simple tensors

$\mathscr{L}_{n,p}^{T}$	fibre of tensor bundle
$\mathscr{L}_n^{2,T}$	tensors over ℓ^2
$\mathscr{P}_k[F,G]$, $\mathscr{Q}_k[F,H]$	Grammian matrices
$\mathscr{R}[.]$	range of [.]
$\mathscr{A}, \mathscr{B}, \mathscr{C}, \mathscr{D}, \mathscr{X}, \mathscr{Y}$	subspaces of $\mathbb{R}^n$
θ	orthogonal difference
$\mathscr{H}$	subspace of $\mathscr{F}(M)$
$\mathscr{S}(x)$	ideal in $\mathscr{L}$
$\mathscr{A}_i$	tensor operator
$\rightharpoonup$	weak convergence
$\mathscr{F}$	Fréchet derivative
$\tilde{\mathscr{H}}$	abstract group of a Lie group
$\hat{i}$ or $\hat{\imath}$	means the component with value i is omitted
$\tilde{\mathbb{R}}$	$\mathbb{R}$ with a special structure
$f_\sim$	induced map of f under relation $\sim$
$\vee$	disjoint union
$\mathfrak{g}_\Delta$	direct sum of root spaces

CONTENTS

1 MATHEMATICAL APPARATUS

1.1 DIFFERENTIABLE MANIFOLDS

1.1.1 Differentiable manifolds

We shall first recall some elementary notions from topology. A *topological space* $(X, \mathcal{O})$ is a set X, together with a set $\mathcal{O}$ of subsets of X which satisfies the properties

(a) $\emptyset, X \in \mathcal{O}$
(b) if $X_1, X_2 \in \mathcal{O}$ then $X_1 \cap X_2 \in \mathcal{O}$
(c) if $\{X_\alpha\}_{\alpha \in A} \subseteq \mathcal{O}$ then $\bigcup_{\alpha \in A} X_\alpha \in \mathcal{O}$.

The elements of $\mathcal{O}$ are called *open subsets* of X. A *neighbourhood* of a point $x \in X$ is a set N such that $x \in Y \subseteq N$ for some $Y \in \mathcal{O}$. The space X is called a *Hausdorff space* if $x_1, x_2 \in X$, $x_1 \neq x_2$ imply that there exist $X_1, X_2 \in \mathcal{O}$ with $x_1 \in X_1$, $x_2 \in X_2$ and $X_1 \cap X_2 = \emptyset$. An *open cover* of a subset $Y \subseteq X$ is a set $\mathcal{O}_1 \subseteq \mathcal{O}$ such that $Y \subseteq \bigcup \mathcal{O}_1$. The subset Y is *compact* if every open cover has a finite subcover.

A function $f: X \to Y$ between topological spaces X and Y is *continuous* (at $x \in X$) if $f^{-1}(W)$ is open for each open set W in Y containing $f(x)$. The function f is a *homeomorphism* if it is bijective and f and f^{-1} are continuous.

A *metric space* (X, d) is a set X together with a *distance function* $d: X \times X \to \mathbb{R}^+$ such that

(a) $d(x, y) = 0$ if and only if $x = y$
(b) $d(x, y) = d(y, x)$ for all $x, y \in X$
(c) $d(x, y) \leqslant d(x, z) + d(z, y)$ for all $x, y, z \in X$.

The *open ball* in X with centre x and radius ε is the set

$$B_\varepsilon(x) = \{y \in X: d(x, y) < \varepsilon\}.$$

Open sets in a metric space are unions of open balls and thus a metric space is a topological space.

The *distance* between a set Y and a point x in a metric space is defined as

$$d(x, Y) = \inf_{y \in Y} \{d(x, y)\}.$$

The *distance* between subsets Y, Z is defined by

$$d(Y, Z) = \max\{\sup_{y \in Y} d(y, Z), \quad \sup_{z \in Z} d(Y, z)\}.$$

Note that the set $(2^X, d)$ of all subsets of X with this distance function is not a metric space since

$$d(B_\varepsilon(x), \overline{B_\varepsilon(x)}) = 0$$

for any ball $B_\varepsilon(x) \triangleq \{y \in X: d(x, y) < \varepsilon\}$. However, if we define the equivalence relation $\sim$ on 2^X by

$$Y \sim Z \quad \text{if and only if } d(Y, Z) = 0,$$

then $(2^X/\sim, d)$ is a metric space, and d is then called the *Hausdorff metric.*

Let M be a topological space. A *chart* on M is a pair (U, ϕ) where $U \subseteq M$ is open and ϕ is a homeomorphism of U onto an open subset of $\mathbb{R}^n$, for some n. If $x^i : \mathbb{R}^n \to \mathbb{R}$ denotes the standard projection on the ith component then, for any chart (U, ϕ), $x^i \circ \phi : U \to \mathbb{R}$ is called the *coordinate function*. We shall often write $x^i \circ \phi$ simply as x^i; then we are effectively identifying U with an open set in $\mathbb{R}^n$ via the homeomorphism ϕ. Two charts $\phi_1 : U_1 \to \mathbb{R}^n$, $\phi_2 : U_2 \to R^n$ are *compatible* if the mapping

$$\phi_2 \circ \phi_1^{-1} : \phi_1(U_1 \cap U_2) \to \phi_2(U_1 \cap U_2)$$

is of class C^∞ (i.e. all partial derivatives exist and are continuous). Of course, if $U_1 \cap U_2 = \emptyset$, then the charts are trivially compatible.

An *atlas* for the topological space M is a collection of compatible charts (U_i, ϕ_i) such that $\bigcup_i U_i = M$. An atlas is *complete* if it cannot be included in a larger atlas.

A *differentiable manifold* (of class C^∞) is a Hausdorff space with a complete atlas of charts. It is possible to specify a differentiable structure on M by defining any C^∞ atlas on M compatible with the given complete atlas, since the completions of both atlases are the same. Any C^∞ atlas on M will therefore define a *differentiable structure.*

It is easy to see that the number n must be constant on each connected component of M. If n is constant on the whole of M, then n is called the *dimension* of the differentiable manifold.

REMARK If we consider C^r mappings throughout the above discussion, we obtain C^r manifolds rather than C^∞ ones. Similarly, if $\mathbb{R}^n$ is replaced by $\mathbb{C}^n$

and we consider mappings which are holomorphic then we obtain *analytic manifolds* (of class C^∞). An atlas then defines the *analytic structure.*

Examples

1. $\mathbb{R}^n$ with the identity chart $id: \mathbb{R}^n \to \mathbb{R}^n$ is a differentiable manifold of dimension n.
2. The $(n-1)$-sphere $S^{n-1} \subseteq \mathbb{R}^n$ defined by $S^{n-1} = \{x \in \mathbb{R}^n : \|x\| = 1\}$ is a differentiable manifold of dimensions $n-1$, whose differentiable structure can be defined by two charts. (Note that no compact manifold can be specified by a single chart.)
3. Given two manifolds M, N with differentiable structures $\{(U_i, \phi_i)\}$, $\{(V_j, \psi_j)\}$ we can define a differentiable structure on the topological Cartesian product $M \times N$ as the collection of product charts $\{(U_i \times V_j, \phi_i \times \psi_j)\}$, these clearly being compatible. For example, the product of two circles $T^2 = S^1 \times S^1$ defines a torus; more generally, $T^n = S^1 \times \cdots \times S^1$ (n times) is an n-dimensional torus.
4. The set of $n \times m$ matrices $M_{n \times m}$ is a differentiable manifold with a single chart $\phi: M_{n\times m} \to \mathbb{R}^{n\times m}$ defined by $\phi(A) = (a_{ij})$, where A is the matrix (a_{ij}).

1.1.2 Differentiable functions

If M and N are differentiable manifolds of dimensions m and n, respectively, and $f: M \to N$ is a given function, then we say that f is *differentiable*‡ at $x \in X$ if the function

$$F \triangleq \psi \circ f \circ \phi^{-1} : \mathbb{R}^m \to \mathbb{R}^n$$

is differentiable for any charts (U, ϕ) of M and (V, ψ) of N where $x \in U, f(x) \in V$. The function $\psi \circ f \circ \phi^{-1}$ is defined on $\phi(f^{-1}(V) \cap U)$ (see Fig. 1.1). We say that f is *differentiable* if it is differentiable at each $x \in X$. Note that the definition of differentiability is independent of the choice of the coordinates ϕ, ψ, since if

$$\tilde{F} \triangleq \tilde{\psi} \circ f \circ \tilde{\phi}^{-1}$$

is another expression for f with respect to the coordinates ϕ, ψ, then the function

$$(\tilde{\psi} \circ \psi^{-1}) \circ F \circ (\phi \circ \tilde{\phi}^{-1})$$

is a restriction of $\tilde{F}$ to an open subset of the domain of F which is differentiable if and only if F is differentiable.

An injection $f: M \to N$ of a differentiable manifold M onto another

‡ If M, N are analytic manifolds, f is called *analytic*.

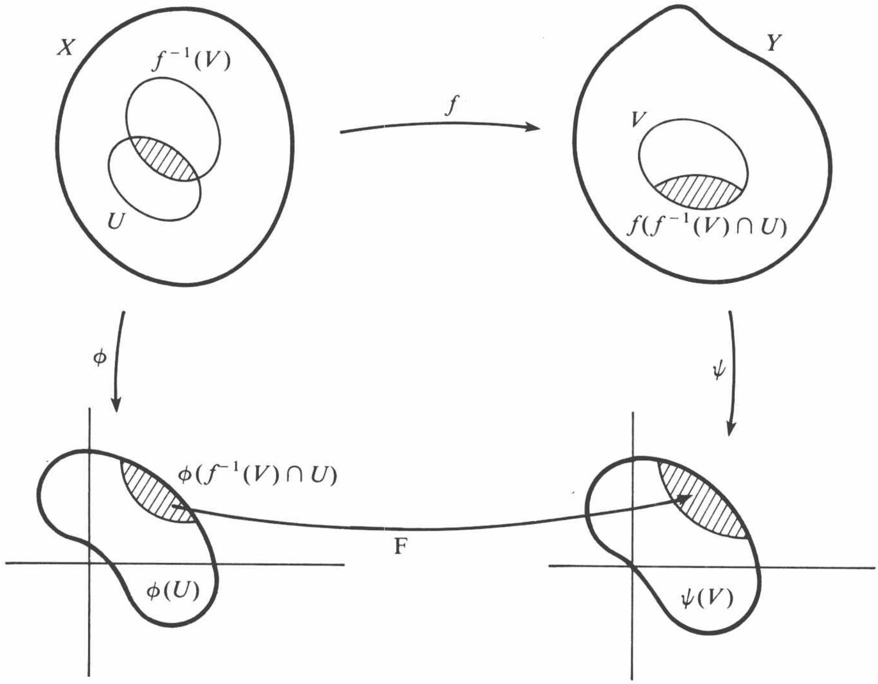

Fig. 1.1 Local representations of functions

differentiable manifold N is called a *diffeomorphism* if f and f^{-1} are differentiable. In this case M and N are said to be *diffeomorphic*.

Examples

1. If $\mathbb{R}^m$, $\mathbb{R}^n$ have their usual differentiable structures, then $f: \mathbb{R}^m \to \mathbb{R}^n$ is differentiable if it is differentiable in the usual sense.
2. Let $\mathbb{R}$ have its usual differentiable structure defined by the identity chart $\phi: \mathbb{R} \to \mathbb{R}, \phi(x) = x$, and let $\tilde{\mathbb{R}}$ be the *set* $\mathbb{R}$ with the differentiable structure defined by the single chart $\tilde{\phi}: \tilde{\mathbb{R}} \to \mathbb{R}$, $\tilde{\phi}(x) = x^3$. Then the identity map $id: \mathbb{R} \to \tilde{\mathbb{R}}$ is not a diffeomorphism since its inverse $id: \tilde{\mathbb{R}} \to \mathbb{R}$ has the coordinate expression $Id(x) \triangleq \phi \circ id \circ \tilde{\phi}^{-1}(x) = x^{1/3}$ which is not differentiable when $x = 0$. However, the map $f = \tilde{\phi}: \tilde{\mathbb{R}} \to \mathbb{R}$ is a diffeomorphism since the coordinate representations of f and its inverse $f^{-1} = (.)^{1/3}$ are both equal to the identity map $id: \mathbb{R} \to \mathbb{R}$.

REMARK Example 2 shows that a topological space can carry two differentiable structures which are not compatible, even though the resulting differentiable manifolds are diffeomorphic. The question then arises as to whether there exist truly different differentiable structures on $\mathbb{R}$. In fact, it turns out that all differentiable structures on $\mathbb{R}$ lead to diffeomorphic manifolds. Even more is true; if $n \neq 4$ then all differentiable structures on $\mathbb{R}^n$ are diffeomorphic. Milnor (1956) showed that there exists an 'exotic' differentiable structure on the seven-sphere S^7 (i.e. one which is not diffeomorphic to the standard structure). A theorem of Donaldson (1983) shows that there exists an exotic structure on $\mathbb{R}^4$; see also Freed and Uhlenbeck (1984).

It is customary to denote the set of differentiable, real-valued functions on a manifold M by $C^\infty(M)$ or $\mathscr{F}(M)$ and those real-valued functions which are defined in a neighbourhood of a point $p \in M$ and are differentiable at p by $C^\infty(p)$ or $\mathscr{F}(p)$. Note that $\mathscr{F}(M)$ (and $\mathscr{F}(p)$) are associative algebras over $\mathbb{R}$ with the operations

$$\begin{aligned} (\alpha f)(p) &= \alpha f(p), && \alpha \in \mathbb{R}, p \in \mathrm{M} \\ (f+g)(p) &= f(p) + g(p), && p \in \mathrm{M} \\ (fg)(p) &= f(p)g(p), && p \in \mathrm{M}, \end{aligned}$$

for all $f, g \in \mathscr{F}(M)$.

1.1.3 Submanifolds

Let M and N be differentiable manifolds of dimensions m and n and let $f\colon M \to N$ be a differentiable mapping. Then we say that f is an *immersion* if for each $p \in M$ there is a neighbourhood U of p in M and a chart (V, ψ) containing $f(p)$ in N such that $\phi \triangleq \psi \circ f | U$ is a chart for M. The manifold M is then said to be *immersed* in N. An injective immersion is called an *embedding*. Since an immersion is clearly locally injective, it follows that an immersion is a local embedding.

A subset M of a manifold N is called a *submanifold* if the canonical injection $i\colon M \subseteq N$ is an embedding, for a given differentiable structure on M.

Examples

1. If M is an open subset of N, then M is a submanifold, and is called an *open submanifold*. Charts on M are just the restrictions of charts on N to M, and it is clear that the manifold dimensions of M and N are the same.
2. Let $\mathbb{R}^1$ and $\mathbb{R}^2$ have their usual differentiable structures and let $f_i\colon \mathbb{R}^1 \to \mathbb{R}^2$, $i = 1, 2, 3$ be the mappings shown (by their images in $\mathbb{R}^2$) in Fig. 1.2. Then f_1 is an immersion and f_2 is an embedding, so that $f_2(\mathbb{R}^1)$ (with the differentiable structure induced from $\mathbb{R}^1$) is a submanifold of $\mathbb{R}^2$. Note, however, that the topology of $f_2(\mathbb{R}^1)$ is induced from $\mathbb{R}^1$ and not that induced as a subspace of $\mathbb{R}^2$, so that any $\mathbb{R}^2$-neighbourhood of p contains points in $f_2(U)$ and $f_2(V)$, where U is

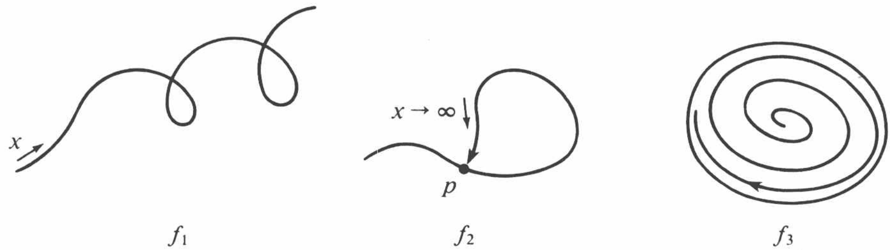

Fig. 1.2 Immersed submanifolds

a neighbourhood of $f_2^{-1}(p)$ and V is a neighbourhood of ∞ in $\bar{\mathbb{R}}^1$ such that $U \cap V = \emptyset$. $f_3 : \mathbb{R}^1 \to \mathbb{R}^2$ is topologically an unstable trajectory of a system with a limit cycle, and is an embedding.

Embedded submanifolds can be characterized as follows:

Theorem 1.1.1
A manifold M is an m-dimensional submanifold of the n-dimensional manifold N if for all $p \in X$, there exists a chart (V, ψ) of N with $p \in V$ such that

(a) $\psi(p) = 0$
(b) the set $W = \{q \in V: y^{m+1} \circ \psi(q) = \cdots = y^n \circ \psi(q) = 0\}$ and the restrictions of $y^1, \ldots, y^m$ to W form a chart of M with $p \in W$. (Here, $(y^1, \ldots, y^n)$ are the coordinate functions.)

Moreover, if f: $M \to N$ is injective and differentiable, and for every $p \in X$ there exists a chart (U, ϕ) of p in M and a chart (V, ψ) of $f(p)$ in N such that the linear map

$$D(\psi \circ f \circ \phi^{-1})(\phi(p)) : \mathbb{R}^m \to \mathbb{R}^n$$

is injective, then $f(M)$ is a submanifold of N with the differentiable structure which makes f: $M \to f(M)$ a diffeomorphism. □

Using the implicit function theorem, it can be shown that if f: $\mathbb{R}^n \to \mathbb{R}$ is a differentiable function and $M = f^{-1}(0)$, then M has a uniquely determined differentiable structure making it an $(n-1)$-dimensional submanifold of $\mathbb{R}^n$ if, for each $p \in M$, $(\text{grad } f)(p) \neq 0$. This can be generalized to functions f: $M \to N$ for manifolds M and N of dimensions m and n $(m \geqslant n)$:

Theorem 1.1.2
If f: $M \to N$ is differentiable and $q \in N$, then $f^{-1}(q)$ is a submanifold of M of dimension $(m - n)$ (or is empty) if for any $p \in f^{-1}(q)$ there are charts