ESSAYS ON MATHEMATICAL EDUCATION

BY

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ITH AN INTRODUCTION BY

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INTRODUCTION

It has always been hard for people to judge with any accuracy the work of their own age, and it is hard for us to do so to-day. In spite of our optimism and of our certainty that we are progressing, what we conceive to be an era of great educational awakening may appear to the historian of the future as one in which noble ideals were sacrificed to the democratizing of the school, and the twentieth century may not rank with the sixteenth when the toll is finally taken.

It is, therefore, with some hesitancy that we should assert that we live in a period of remarkable achievement in all that pertains to education. That the period is one of advance is in harmony with the general principle of evolution, but that all that we do is uniformly progressive is not at all in accord with general experience. Certain it is that the present time is one of agitation, of the shattering of idols, and of the setting up of strange gods in their places. Nothing is sacred to the iconoclast, and he is found in the school as he is found in the church, in government, and in the social world.

Among the objects of attack in this generation is "the science venerable" that has come down to us from Pythagoras and Euclid, from Mohammed ben Musa and Bhaskara, and from Cardan, Descartes, and Newton. And yet it does not seem to be mathematics itself that is challenged so much as the way in which it has been presented to the youth in our schools, and to most of us the challenge seems justified. With all the excellence of Euclid, his work is not for the child; and with all the value of formal algebra, the science needs some other introduction than the arid one until recently accorded to it.

It is on this account that Mr. Carson's work in the English schools and before bodies of English teachers has great value. He is thoroughly trained as a mathematician, is a product of the college where Newton studied and taught, is a lover of the science in its purest form, and has had an unusual amount of experience in the technical applications of the subject; but he is a teacher by instinct and by profession, and is imbued with the feeling that mathematics can be saved to the school only through an improvement in our methods of teaching and in our selection of material. He stands for the principle that mathematics must be made to appeal to the learner as interesting and valuable, and he has shown in his own classes that, after this appeal has been successful, pupils need to be held back rather than driven forward in this branch of learning.

It is because of this feeling on the part of Mr. Carson that his essays on the teaching of mathematics have peculiar value at this time. They will encourage teachers to continue their advocacy of a worthy form of mathematics, at the same time seeking better lines of approach and endeavouring to relate the subject in a reasonable manner to the various other interests of the pupil. The problem is much the same everywhere, but the ties of a common language, a common spirit of freedom, and a common ancestry make it practically identical in English-speaking lands. On this account we, in the United States, feel that Mr. Carson's message is quite as much to us as to his own countrymen, and we shall appreciate it as we have appreciated the noteworthy work that he has already achieved in the teaching of mathematics in England.

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SOME PRINCIPLES OF MATHEMATICAL EDUCATION

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SOME PRINCIPLES OF MATHEMATICAL EDUCATION

Of all the problems which have perplexed teachers of mathematics in this generation, probably none has been more irritating and insistent than the choice of assumptions which must be made in each branch of the science. In geometry, in analysis, in mechanics, one and the same difficulty arises. Are we to prove that any two sides of a triangle are greater than the third? That the limit of the sum of a finite number of functions is equal to the sum of their limits? That the total momentum of two bodies is uninfluenced by their mutual action? And in every such case, on what is the proof to depend? A clear understanding of the answers to such questions, or, better still, a clear understanding of principles by which answers may be found, would go far to co-ordinate and simplify elementary teaching; the object of this paper is to state such principles and indicate their application.

Axiom, Postulate, Proof

It is first necessary to lay down definitions, as precise as may be possible, of the terms "axiom," "postulate," "proof." It is not implied that these definitions should be insisted on, or the terms used, in elementary teaching; nothing could be more likely to lead to failure. But a full comprehension of each is essential to every teacher of mathematics, and is too often lacking in current usage.

An axiom, or "common notion" in Euclid's language, is a statement which is true of all processes of thought, whatever be the subject matter under discussion. Thus the following are axioms: "If *A* is identical with *B*, and *C* is different from *B*, then *C* is different from *A*." "If *B* is a necessary consequence of *A*, and also *C* of *B*, then *C* is a necessary consequence of *A*." But "Two and two make four" and "The straight line is the shortest distance between two points" are not axioms, although they may be considered no less obvious. A statement is not an axiom because it is obvious, but because it concerns universal forms of thought, and not a particular subject matter such as arithmetic, geometry, and the like.

A postulate is a statement which is assumed concerning a particular subject matter; for example, "The whole is greater than a part" (subject matter, finite aggregates); "All right angles are equal" (subject matter, Euclidean space). It is essential to observe that, whereas an axiom is an axiom once for all, a postulate in one treatment of a science may not be a postulate in another. In Euclid's development of geometry, the statement that any two sides of a triangle are together greater than the third side is not a postulate, because it is deduced from other statements (postulates) which are avowedly assumed; but in many current developments it is adopted at once, without reference to other statements, and is therefore a postulate in such cases. To use an unconventional but expressive term, postulates are "jumping-off places" for the logical exploration of a subject. Their number and nature are immaterial; they may be readily acceptable, or difficult of credence. Their one function is to supply a basis for reasoning, which is conducted in accordance with the axioms. Postulates are thus doubly relative: they relate to one particular subject matter (number, space, and so on) and to one particular method of viewing that subject matter.

A statement which is deduced, by use of the axioms, from two or more postulates is said to be proved. There is thus no such thing as absolute proof. Proofs are related to the postulates on which they are based, and a demand for a proof must inevitably be met by a counter demand for a place to start from, that is, for some postulates. When a statement is said to have been proved, what is meant is that it has been shown to be a logical consequence of some other statements which have been accepted; if these statements are found to be incorrect, the statement which is said to be proved can no longer be accepted, though the logical character of the proof is in no way impugned. Thus the type of a proof is, "If *A*, then *B*"; relentless and final certainty surrounds "then"; but *A*, which is assumed in the "if," may nevertheless be utterly fantastic as viewed in the light of experience.

The Three Functions of Mathematics

The first application of mathematics to any domain of knowledge can now be explained. Starting from postulates, the truth of which is no concern of mathematics, sets of deductions are evolved by use of the axioms; agreement of the results with experience strengthens the evidence in favour of these postulates. If this evidence be deemed sufficient, as, for example, in geometry and mechanics, then deduction yields acceptable results which could not otherwise have been predicted or ascertained.

It is here that the prevailing concept of the power of mathematics ends; but such a concept presents a view of the subject so limited and distorted as to be almost grotesque. The process just described may be regarded as an upward development; a downward research is also possible, and no less valuable. It consists of a logical review of the set of postulates which have been adopted; in the result, either it is shown that some must be rejected, or the evidence in favour of all may be considerably enhanced. This review consists of two processes, which will be described in turn.

It is first necessary to ascertain whether the set of postulates is consistent; that is, whether some among them may not be logically contradictory of others. For example, Euclid defines parallel straight lines as coplanar lines which do not intersect, and proves in his twenty-seventh proposition that such lines can be drawn; for this purpose he uses his fourth postulate, which makes no allusion to parallels. If he had included among his postulates another, stating that every pair of coplanar lines intersect if produced sufficiently far, and had omitted his definition of parallel lines, his postulates would not have been consistent; for the twenty-seventh proposition proves that if the fourth postulate be granted, then the existence of non-intersecting coplanar lines must be admitted also. It is essential to realize that the contradiction implied in the term "inconsistent" is based on logic, not on experience; assumptions which are contrary to all experience are not thereby inconsistent. There is nothing in logic to veto the assumption that, for certain types of matter, weight and mass are inversely proportional; or that life may exist where there is no atmosphere, as on the moon. Such assumptions are not inconsistent with the other postulates of mechanics or biology; they are merely contrary to all experience gained up to the present time.

Here, then, is the second function of the mathematician — the investigation of the consistence of a set of postulates. And the task is not superfluous. Physical measurements are perforce inaccurate, and a set of inconsistent assumptions might well appear to be consistent with actual observations. More accurate measurements must, of course, expose the discrepancy, but these may for ever remain beyond our powers; logic renders them superfluous by demonstrating the consistence or otherwise of each set considered.

The next investigation concerns the redundance of a set of postulates. Such a set is said to be redundant if some of its members are logical consequences of others. For example, any ordinary adult will accept without difficulty the properties of congruent figures, the angle properties of parallel lines, and the properties of similar figures, as in maps or plans, regarding them as "in the nature of things." And electricians may, by experiment, convince themselves first, that Coulomb's law of attraction is very approximately true; and secondly, that within the limits of observation there is no electric force in the interior of a closed conductor. In neither the one case nor the other need there be the least suspicion that the statements are logically connected, so that they must stand or fall together. Yet so it is, and the fact is expressed by the statement that the assumptions are redundant.

The investigation of redundance, and the demonstration that sets of postulates are free therefrom, forms the third function of the mathematician. Its value, in connection with any subject matter to which it may be applied, may not at once be evident. It is based on the fact that all experiments, necessary and inevitable though they be, are nevertheless sources of uncertainty; it reduces this uncertainty to a minimum by removing the redundant assumptions into the category of propositions, and exposing the science in question as based on a minimum of assumption. And more; it can offer several alternative sets of assumptions for choice, that one being taken which is most nearly capable of verification. The labours of Faraday resulted in the offer of such a choice to electricians; either, they were told, you can base electrostatics (*inter alia*) on Coulomb's experiment, or on the absence of electric force inside a closed conductor; it is logically immaterial which course you adopt. The latter experiment, being far more capable of accurate demonstration in the laboratory, is chosen as the primary basis for faith in the deductions of electrostatics — a faith which is, of course, very much strengthened as such deductions are found to accord with our experience. But these considerations are for the physicist; the task of the mathematician is ended when he has put forward, for choice by the physicist, alternative sets of assumptions which are at once consistent with each other and free from redundance. In this way does he free the physicist, so far as may be, from the uncertainties of assumption, and assure him that no further increase of such freedom can be attained.[1]

[1] The antithesis between mathematician and physicist does not imply that the functions are of necessity performed by different individuals; it is used merely to enforce the argument.

Such is the range of application of mathematics to other sciences. When complete it reveals each science as a firmly knit structure of logical reasoning, based on assumptions whose number and nature are clearly exposed; of these assumptions it can be asserted that no one is inconsistent with the others, and that each is independent of the others. There is thus no fear that contradictions may in time emerge, and no false hope that one assumption may in time be shown to be a logical consequence of the others. Finality has been reached.

The acute critic may, of course, ask the mathematician whether his own house is in order. What is the precise statement of the axioms which are the basis of his science, and can they be shown to depend on a set of consistent mental postulates, free from redundance? Here it need only be said that the labours of the last generation have done much to answer these questions, and that their complete solution is certainly possible, if not actually achieved; to go further would be beyond the limits of this paper.

The Didactic Problem

The complete application of mathematics to any branch of knowledge being thus exhibited, the didactic problem can now be stated in explicit terms. In any given science — geometry, mechanics, and so on — what is the right point of entry to the structure, and in what order should its exploration be made? What results should be regarded as postulates, and should their consistence and possible interdependence one on another be investigated before upward deduction is undertaken? Should the minimum number be chosen on the ground that the pupil should at once be placed in possession of the ultimate point of view? Or should some larger number be taken, and if so, on what principles should they be chosen?

Bearing in mind that the pupils concerned are not presumed to be adults, it is easy to indicate principles from which answers to such questions may be deduced. One of the few really certain facts about the juvenile mind is that it revels in exploration of the unknown, but loathes analysis of the known. It is often said that boys and girls are indifferent to, and cannot appreciate, exact logic; that it is unwise to force detailed reasoning upon them. Few statements are farther from the truth. Logic, provided that it leads to a comprehensible goal, is not only appreciated, but demanded, by pupils whose instincts are normal. But the goal must be comprehensible; it must not be a result as easily perceived as the assumptions on which the proof is based. Let any one with experience in examining consider the types of answer given to two problems; one, an "obvious" rider on congruence, involving possibly the pitfall of the ambiguous case; the other, some simple but not obvious construction or rider concerning areas or circles. In the former, paper after paper exhibits fumbling uncertainty or bad logic; in the latter, there is usually success or silence, and more usually success; bad logic is hardly ever found. The phenomenon is too universal to be comfortably accounted for by abuse of the teachers; the abuse must be transferred to the crass methods which enforce the premature application of logic to analysis of the known, rather than to exploration of the unknown.

The natural order of exploration should now be evident. Let the leading results of the science under consideration be divided into two groups: one, those which are acceptable, or can be rendered acceptable by simple illustration, to the pupils under consideration; the other, those which would never be suspected and whose verification by experiment would at once produce an unreal and artificial atmosphere. Let the former group — which in geometry would include many of Euclid's propositions — be adopted as postulates, and let deductions be made from them with full rigour. Wherever possible, let the results of such deductions be tested by experiment, so as to give the utmost feeling of confidence in the whole structure. Later, when speculation becomes more natural, let it be suggested that gratuitous assumption is perhaps inadvisable, and let the meanings of the consistence and redundance of the set of postulates be explained. Finally, if it prove possible, let the postulates be analysed, their consistence and independence be demonstrated, and the science exposed in its ultimate form.

These second and third stages are even more essential to a "liberal education" than the first, for they exhibit scientific method and human knowledge in their true aspect. It is not suggested that they can be dealt with in schools, except perhaps tentatively in the last year of a long course. But it is definitely asserted that the general ideas involved should form part of the compulsory element of every University course, even though details be excluded, for they are of the very essence of the spirit of mathematics. The method of developing such ideas remains to be considered.

It may be presumed that the pupils concerned have some knowledge of arithmetic, geometry, the calculus, and mechanics, each subject having been developed from a redundant set of postulates. In which, then, of these four branches is it most natural to suggest the analysis of these assumptions?

Since analysis of the known may still be presumed to have its dangers, the branch chosen must be that one in which the investigation bears this aspect least prominently. Now the main ideas of arithmetic, geometry, and the calculus are so firmly held by boys and girls, that any attempt to discuss them in detail produces revolt or boredom. Such attempts account for much; the writer can well remember his feelings on first seeing a formal proof that the sum of a definite number of continuous functions is itself a continuous function; and at the same time he realized to the full that the proposition might well be untrue if the number of functions were not finite. Ground such as this is unfavourable for the development of this new analysis.

The same is by no means true of mechanics. Here the postulates, acceptable though they be, have been elucidated within the memory of the pupils, and they may reasonably be asked to examine the facility with which these assumptions were made, and to consider whether the evidence can in any way be strengthened. This being done, the ideas of consistence and redundance can be developed, and some idea of the structure of a science imparted. Even then it may probably be wise to lay little stress on analysis of the geometrical postulates; if the ideas are realized in connection with mechanics, we may well leave the seed to mature in minds to which it is congenial.

In the view of mathematics here taken, its various branches are regarded as structures with many possible entrances, and the discussion has been concerned with the