INTRODUCTORY LINEAR ALGEBRA with APPLICATIONS

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To my children, Lisa and Stephen

Preface

Linear algebra is fast becoming a standard part of the undergraduate mathematical training of a diverse number of students for at least two reasons. First, few subjects can claim to have such widespread applications in other areas of mathematics—multivariable calculus, differential equations, and probability theory, for example—as well as in physics, biology, chemistry, economics, psychology, sociology, and all fields of engineering. Second, the subject provides the student with his first introduction to postulational or axiomatic mathematics.

In a semester or quarter course in linear algebra the instructor and student are frequently faced with several problems. The important subject of eigenvalues is often studied in too much haste to do the material justice. The student seldom sees any meaningful applications of linear algebra, and he rarely gets any feel for the widespread use, in conjunction with computers, of the new area of numerical linear algebra. In this book we have tried to deal with these problems.

This book presents a brief introduction to linear algebra and to some of its significant applications and is designed for a semester or quarter course at the freshman or sophomore level. If most of the applications are covered, there is enough material for a year course. Calculus is not a prerequisite.

The emphasis is on the computational and geometrical aspects of the subject, keeping the abstraction down to a minimum. Thus we sometimes omit proofs of difficult or less rewarding theorems, while amply illustrating them with examples. The proofs that are included are presented at a level appropriate for the student. We have also concentrated our attention on the essential areas of linear algebra; the book does not attempt to cover the subject exhaustively.

Chapter 1 deals with matrices and their properties; Chapter 2 covers determinants in a brief manner. In Chapter 3, on vectors and vector spaces, and Chapter 4, on linear transformations and matrices, we explore some of the geometric ideas of the subject. Although the general definition of a vector space is given and the results are stated in terms of vector spaces, all examples considered are n-space and its subspaces. Thus R^n is viewed as a generalization of R^2 and R^3, the familiar 2- and 3-spaces. Chapter 5, on eigenvalues and eigenvectors, settles the diagonalization problem for symmetric matrices. Chapter 6 contains an introduction to linear programming, an extremely important application of linear algebra. Chapter 7 covers several other diverse applications of linear algebra: lines and planes, quadratic forms, graph theory, the theory of games, least squares, and linear economic models. This material includes applications to the social sciences and to economics. In Chapter 8, on numerical linear algebra, we deal rather briefly with numerical methods commonly used in linear algebra to solve linear systems of equations and to find eigenvalues and eigenvectors of matrices. These methods are widely used in conjunction with computers. However, the examples and exercises presented here can all be worked with desk or pocket-size calculators.

It would be difficult, at this time, to ignore the use of the computer in linear algebra, since most computations involving linear algebra in real problems are carried out on computers. This relationship is briefly explored in the appendix. First, we provide a list of 53 *computer projects* grouped by sections corresponding to the text. These projects are of widely differing degrees of difficulty, and the student is expected to have adequate programming skills to handle these tasks. Next, we discuss the availability of *canned software* to implement linear algebra techniques. The use of many of these programs requires some programming skill, although some of them only require that the user know how to use a keypunch machine to punch up the data. Finally, we outline some of the features of APL, a programming language that is especially suitable for matrix manipulations and is easy to learn; we also make several remarks about BASIC, a general-purpose computer language that is being widely used with time-sharing systems.

The exercises in this book are grouped into two classes. The first class, "Exercises," contains routine exercises. The second class, "Theoretical Exercises," contains exercises that fill in gaps in some of the proofs and amplify material in the text. These exercises can be used to raise the level of the course and to challenge the more capable and interested student.

For maximum flexibility the applications have all been grouped together as Chapters 6, 7, and 8. These three chapters are almost entirely independent, the one exception being Section 7.4, "The Theory of Games," which requires Chapter 6 as preliminary study. The first five chapters, which constitute the basic linear algebra material, can be comfortably covered in a semester or quarter (see a suggested pace below), and there should be time left over to do some of the applications. These can either be covered after the first five chapters have been studied or they can be taken up after the required material has been developed. The accompanying chart gives the prerequisites for each of the applications. Moreover, some of the applications, in particular those in Chapters 7 and 8, can also be used as independent student projects. The author's classroom experience with the latter approach has met with favorable student reaction. Thus the instructor can be quite selective both in the choice of material and in the method of study of the applications.

Application	Prerequisite
Chapter 6	Section 1.3
Section 7.1	Chapter 3
Section 7.2	Chapter 5
Section 7.3	Section 1.2
Section 7.4	Chapter 6
Section 7.5	Section 1.3
Section 7.6	Chapter 1
Section 8.1	None
Section 8.2	Section 1.3
Section 8.3	Chapter 5

Suggested Pace for Basic Material (Chapters 1–5)

Chapter 1	6 lectures
Chapter 2	4 lectures
Chapter 3	10 lectures (omit Section 3.3)
Chapter 4	5 lectures
Chapter 5	5 lectures
	30 lectures

This schedule, which can be readily modified by spending more time on the theoretical exercises and proofs, emphasizes the essential material in each section.

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B. K.

Contents

1

Linear Equations and Matrices

1.1 Linear Systems

A good many problems in the natural and social sciences as well as in engineering and science deal with equations relating two sets of variables. An equation of the type

$$y = ax,$$

expressing the variable y in terms of the variable x and the constant a, is called a linear equation. The word "linear" is used here because the graph of the above equation is a straight line. Similarly, the equation

$$y = a_1x_1 + a_2x_2 + \cdots + a_nx_n, \tag{1}$$

expressing y in terms of the variables $x_1, x_2, \ldots, x_n$ and the constants $a_1, a_2, \ldots, a_n$, is called a **linear equation**. In many applications we are given y and must find numbers $x_1, x_2, \ldots, x_n$ satisfying (1).

A **solution** to a linear equation (1) is an ordered collection of n numbers $s_1, s_2, \ldots, s_n$ which have the property that (1) is satisfied when $x_1 = s_1$, $x_2 = s_2, \ldots, x_n = s_n$ are substituted in (1).

Thus $x_1 = 2$, $x_2 = 3$, and $x_3 = -4$ is a solution to the linear equation

$$6x_1 - 3x_2 + 4x_3 = -13,$$

because

$$6(2) - 3(3) + 4(-4) = -13.$$

More generally, a **system of *m* linear equations** in ***n* unknowns**, or simply a **linear system**, is a set of m linear equations each in n unknowns. A linear system can be conveniently denoted by

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= y_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= y_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= y_m. \end{aligned} \tag{2}$$

Thus the ith equation is

$$a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n = y_i.$$

In (2) the a_{ij} are known constants. Given values of $y_1, y_2, \ldots, y_n$, we want to find values of $x_1, x_2, \ldots, x_n$ that will satisfy each equation in (2).

A **solution** to a linear system (2) is an ordered collection of n numbers $s_1, s_2, \ldots, s_n$, which have the property that each equation in (2) is satisfied when $x_1 = s_1$, $x_2 = s_2, \ldots, x_n = s_n$ are substituted in (2).

The reader has already had experience in solving linear systems by the method of *elimination*. That is, we eliminate some of the unknowns by adding a multiple of one equation to another equation. Most likely, the reader has confined his earlier work in this area to linear systems in which $m = n$, that is, linear systems having as many equations as unknowns. In this course we shall broaden our outlook by dealing with systems in which we have $m = n$, $m < n$, and $m > n$. Indeed, there are numerous applications in which $m \neq n$. If we deal with two, three, or four unknowns, we shall often write them as x, y, z, and w.

Example 1. Consider the linear system

$$\begin{aligned} x - 3y &= -3 \\ 2x + y &= 8. \end{aligned} \tag{3}$$

To eliminate x, we subtract twice the first equation from the second, obtaining

$$7y = 14,$$

an equation having no x term. We have eliminated the unknown x. Then solving for y, we have

$$y = 2,$$

and substituting into the first equation of (3), we obtain

$$x = 3.$$

Then $x = 3, y = 2$ is the only solution to the given linear system.

Example 2. Consider the linear system

$$\begin{aligned} x - 3y &= -7 \\ 2x - 6y &= 7. \end{aligned} \tag{4}$$

Again, we decide to eliminate x. We subtract twice the first equation from the second one, obtaining

$$0 = 21,$$

which makes no sense. This means that (4) has no solution. We might have come to the same conclusion from observing that in (4) the left side of the second equation is twice the left side of the first equation, but the right side of the second equation is not twice the right side of the first equation.

Example 3. Consider the linear system

$$\begin{aligned} x + 2y + 3z &= 6 \\ 2x - 3y + 2z &= 14 \\ 3x + y - z &= -2. \end{aligned} \tag{5}$$

To eliminate x, we subtract twice the first equation from the second one and three times the first equation from the third one, obtaining

$$\begin{aligned} -7y - 4z &= 2 \\ -5y - 10z &= -20. \end{aligned} \tag{6}$$

This is a system of two equations in the unknowns y and z. We divide the second equation of (6) by -5, obtaining

$$\begin{aligned} -7y - 4z &= 2 \\ y + 2z &= 4, \end{aligned}$$

which we write, by interchanging equations, as

$$\begin{aligned} y + 2z &= 4 \\ -7y - 4z &= 2. \end{aligned} \tag{7}$$

We now eliminate y in (7) by adding 7 times the first equation to the second one, to obtain

$$10z = 30,$$

or

$$z = 3. \tag{8}$$

Substituting this value of z into the first equation of (7), we find $y = -2$. Substituting these values of y and z into the first equation of (5), we find $x = 1$. We might further observe that our elimination procedure has effectively produced the following linear system:

$$\begin{aligned} x + 2y + 3z &= 6 \\ y + 2z &= 4 \\ z &= 3, \end{aligned} \tag{9}$$

obtained by using the first equations of (5) and (7) as well as (8). The importance of the procedure lies in the fact that the linear systems (5) and (9) have exactly the same solutions. Of course, (9) can be solved quite easily.

Example 4. Consider the linear system

$$\begin{aligned} x + 2y - 3z &= -4 \\ 2x + y - 3z &= 4. \end{aligned} \tag{10}$$

Eliminating x, we subtract twice the first equation from the second equation, to obtain

$$-3y + 3z = 12. \tag{11}$$

We must now solve (11). A solution is

$$y = z - 4,$$

where z can be any real number. Then from the first equation of (10),

$$\begin{aligned} x &= -4 - 2y + 3z \\ &= -4 - 2(z - 4) + 3z \\ &= \quad z + 4. \end{aligned}$$

Thus a solution to the linear system (10) is

$$\begin{aligned} x &= z + 4 \\ y &= z - 4 \\ z &= \text{any real number.} \end{aligned}$$

This means that the linear system (10) has infinitely many solutions. Every time we assign a value to z we obtain another solution to (10). Thus, if $z = 1$, then

$$x = 5, \quad y = -3, \quad \text{and} \quad z = 1$$

is a solution, while if $z = -2$, then

$$x = 2, \quad y = -6, \quad \text{and} \quad z = -2$$

is another solution.

Example 5. Consider the linear system

$$\begin{aligned} x + 2y &= 10 \\ 2x - 2y &= -4 \\ 3x + 5y &= 26. \end{aligned} \tag{12}$$

Eliminating x, we subtract twice the first equation from the second one to obtain $-6y = -24$, or

$$y = 4.$$

Subtracting three times the first equation from the third one, we obtain $-y = -4$, or

$$y = 4.$$

Thus we have been led to the system

$$\begin{aligned} x + 2y &= 10 \\ y &= 4 \\ y &= 4, \end{aligned} \tag{13}$$

which has the same solutions as (12). Substituting $y = 4$ in the first equation of (13), we obtain $x = 2$. Hence $x = 2, y = 4$ is a solution to (12).

Example 6. Consider the linear system

$$\begin{aligned} x + 2y &= 10 \\ 2x - 2y &= -4 \\ 3x + 5y &= 20. \end{aligned} \tag{14}$$

To eliminate x, we subtract twice the first equation from the second one to obtain $-6y = -24$, or

$$y = 4.$$

Subtracting three times the first equation from the third one, we obtain $-y = -10$, or

$$y = 10.$$

Thus we have been led to the system

$$\begin{aligned} x + 2y &= 10 \\ y &= 4 \\ y &= 10, \end{aligned} \tag{15}$$

which has the same solutions as (14). Since (15) has no solutions, we conclude that (14) has no solutions.

We have seen that the method of elimination consists of repeatedly performing the following operations:

1. Interchange two equations.
2. Multiply an equation by a nonzero constant.
3. Add a multiple of one equation to another equation.