Lecture Notes in Mathematics

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Hartmut Führ

Abstract Harmonic Analysis of Continuous Wavelet Transforms

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Preface

This volume discusses a construction situated at the intersection of two different mathematical fields: Abstract harmonic analysis, understood as the theory of group representations and their decomposition into irreducibles on the one hand, and wavelet (and related) transforms on the other. In a sense the volume reexamines one of the roots of wavelet analysis: The paper [60] by Grossmann, Morlet and Paul may be considered as one of the initial sources of wavelet theory, yet it deals with a unitary representation of the affine group, citing results on discrete series representations of nonunimodular groups due to Duflo and Moore. It was also observed in [60] that the discrete series setting provided a unified approach to wavelet as well as other related transforms, such as the windowed Fourier transform.

We consider generalizations of these transforms, based on a representation-theoretic construction. The construction of continuous and discrete wavelet transforms, and their many relatives which have been studied in the past twenty years, involves the following steps: Pick a suitable basic element (the *wavelet*) in a Hilbert space, and construct a system of vectors from it by the action of certain prescribed operators on the basic element, with the aim of expanding arbitrary elements of the Hilbert space in this system. The associated *wavelet transform* is the map which assigns each element of the Hilbert space its expansion coefficients, i.e. the family of scalar products with all elements of the system. A *wavelet inversion formula* allows the reconstruction of an element from its expansion coefficients.

Continuous wavelet transforms, as studied in the current volume, are obtained through the action of a group via a unitary representation. Wavelet inversion is achieved by integration against the left Haar measure of the group. The key questions that are treated –and solved to a large extent– by means of abstract harmonic analysis are: Which representations can be used? Which vectors can serve as wavelets?

The representation-theoretic formulation focusses on one aspect of wavelet theory, the inversion formula, with the aim of developing general criteria and providing a more complete understanding. Many other aspects that have made wavelets such a popular tool, such as discretization with fast algorithms and the many ensuing connections and applications to signal and image processing, or, on the more theoretical side, the use of wavelets for the characterization of large classes of function spaces such as Besov spaces, are lost when we move on to the more general context which is considered here. One of the reasons for this is that these aspects often depend on a specific *realization* of a representation, whereas abstract harmonic analysis does not differentiate between unitarily equivalent representations.

In view of these shortcomings there is a certain need to justify the use of techniques such as direct integrals, entailing a fair amount of technical detail, for the solution of problems which in concrete settings are often amenable to more direct approaches. Several reasons could be given: First of all, the inversion formula is a crucial aspect of wavelet and Gabor analysis. Analogous formulae have been – and are being – constructed for a wide variety of settings, some with, some without a group-theoretic background. The techniques developed in the current volume provide a systematic, unified and powerful approach which for type I groups yields a complete description of the possible choices of representations and vectors. As the discussion in Chapter 5 shows, many of the existing criteria for wavelets in higher dimensions, but also for Gabor systems, are covered by the approach.

Secondly, Plancherel theory provides an attractive theoretical context which allows the unified treatment of related problems. In this respect, my prime example is the discretization and sampling of continuous transforms. The analogy to real Fourier analysis suggests to look for nonabelian versions of Shannon's sampling theorem, and the discussion of the Heisenberg group in Chapter 6 shows that this intuition can be made to work at least in special cases. The proofs for the results of Chapter 6 rely on a combination of direct integral theory and the theory of Weyl-Heisenberg frames. Thus the connection between wavelet transforms and the Plancherel formula can serve as a source of new problems, techniques and results in representation theory.

The third reason is that the connection between the initial problem of characterizing wavelet transforms on one side and the Plancherel formula on the other is beneficial also for the development and understanding of Plancherel theory. Despite the close connection, the answers to the above key questions require more than the straightforward application of known results. It was necessary to prove new results in Plancherel theory, most notably a precise description of the scope of the pointwise inversion formula. In the nonunimodular case, the Plancherel formula is obscured by the *formal dimension operators*, a family of unbounded operators needed to make the formula work. As we will see, these operators are intimately related to *admissibility conditions* characterizing the possible wavelets, and the fact that the operators are unbounded has rather surprising consequences for the existence of such vectors. Hence, the drawback of having to deal with unbounded operators, incurring the necessity to check domains, turns into an asset.

Finally the study of admissibility conditions and wavelet-type inversion formulae offers an excellent opportunity for getting acquainted with the Plancherel formula for locally compact groups. My own experience may serve as an illustration to this remark. The main part of the current is concerned with the question how Plancherel theory can be employed to derive admissibility criteria. This way of putting it suggests a fixed hierarchy: First comes the general theory, and the concrete problem is solved by applying it. However, for me a full understanding of the Plancherel formula on the one hand, and of its relations to admissibility criteria on the other, developed concurrently rather than consecutively. The exposition tries to reproduce this to some extent. Thus the volume can be read as a problem-driven – and reasonably self-contained– introduction to the Plancherel formula.

As the volume connects two different fields, it is intended to be open to researchers from both of them. The emphasis is clearly on representation theory. The role of group theory in constructing the continuous wavelet transform or the windowed Fourier transform is a standard issue found in many introductory texts on wavelets or time-frequency analysis, and the text is intended to be accessible to anyone with an interest in these aspects. Naturally more sophisticated techniques are required as the text progresses, but these are explained and motivated in the light of the initial problems, which are existence and characterization of admissible vectors. Also, a number of well-known examples, such as the windowed Fourier transform or wavelet transforms constructed from semidirect products, keep reappearing to provide illustration to the general results. Specifically the Heisenberg group will occur in various roles.

A further group of potential readers are mathematical physicists with an interest in generalized coherent states and their construction via group representations. In a sense the current volume may be regarded as a complement to the book by Ali, Antoine and Gazeau [1]: Both texts consider generalizations to the discrete series case. [1] replaces the square-integrability requirement by a weaker condition, but mostly stays within the realm of irreducible representations, whereas the current volume investigates the irreducibility condition. Note however that we do not comment on the relevance of the results presented here to mathematical physics, simply for lack of competence.

In any case it is only assumed that the reader knows the basics of locally compact groups and their representation theory. The exposition is largely self-contained, though for known results usually only references are given. The somewhat introductory Chapter 2 can be understood using only basic notions from group theory, with the addition of a few results from functional and Fourier analysis which are also explained in the text. The more sophisticated tools, such as direct integrals, the Plancherel formula or the Mackey machine, are introduced in the text, though mostly by citation and somewhat concisely. In order to accomodate readers of varying backgrounds, I have marked some of the sections and subsections according to their relation to the core material of the text. The core material is the study of admissibility conditions, discretization and sampling of the transforms. Sections and subsections with the superscript * contain predominantly technical results and arguments which are indispensable for a rigorous proof, but not necessarily for an understanding and assessment of results belonging to the core material. Sections and subsections marked with a superscript ** contain results which may be considered diversions, and usually require more facts from representation theory than we can present in the current volume. The marks are intended to provide some orientation and should not be taken too literally; it goes without saying that distinctions of this kind are subjective.

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Neuherberg, December 5, 2004 *Hartmut Führ*

Contents

1

Introduction

1.1 The Point of Departure

In one of the papers initiating the study of the continuous wavelet transform on the real line, Grossmann, Morlet and Paul [60] considered systems $(\psi_{b,a})_{b,a \in \mathbb{R}\times\mathbb{R}'}$ arising from a single function $\psi \in \mathrm{L}^2(\mathbb{R})$ via

$$\psi_{b,a}(x) = |a|^{-1/2}\psi\left(\frac{x-b}{a}\right) \quad .$$

They showed that every function ψ fulfilling the **admissibility condition**

$$\int_{\mathbb{R}'} \frac{|\widehat{\psi}(\omega)|^2}{|\omega|} d\omega = 1 \quad , \tag{1.1}$$

where $\mathbb{R}' = \mathbb{R} \setminus \{0\}$, gives rise to an **inversion formula**

$$f = \int_{\mathbb{R}} \int_{\mathbb{R}'} \langle f, \psi_{b,a} \rangle \psi_{b,a} \frac{da}{|a|^2} db \quad , \tag{1.2}$$

to be read in the weak sense. An equivalent formulation of this fact is that the **wavelet transform**

$$f \mapsto V_\psi f \quad , \quad V_\psi f(b,a) = \langle f, \psi_{b,a} \rangle$$

is an isometry $\mathrm{L}^2(\mathbb{R}) \to \mathrm{L}^2(\mathbb{R} \times \mathbb{R}', db\frac{da}{|a|^2})$. As a matter of fact, the inversion formula was already known to Calderón [27], and its proof is a more or less elementary exercise in Fourier analysis.

However, the admissibility condition as well as the choice of the measure used in the reconstruction appear to be somewhat obscure until read in group-theoretic terms. The relation to groups was pointed out in [60] –and in fact earlier in [16]–, where it was noted that $\psi_{b,a} = \pi(b,a)\psi$, for a certain representation π of the affine group G of the real line. Moreover, (1.1) and (1.2) have natural group-theoretic interpretations as well. For instance, the measure used for reconstruction is just the left Haar measure on G.

Hence, the wavelet transform is seen to be a special instance of the following construction: Given a (strongly continuous, unitary) representation $(\pi, \mathcal{H}_\pi)$ of a locally compact group G and a vector $\eta \in \mathcal{H}_\pi$, we define the **coefficient operator**

$$V_\eta : \mathcal{H}_\pi \ni \varphi \mapsto V_\eta \varphi \in C_b(G) \quad , \quad V_\eta \varphi(x) = \langle \varphi, \pi(x)\eta \rangle \quad .$$

Here $C_b(G)$ denotes the space of bounded continuous functions on G.

We are however mainly interested in inversion formulae, hence we consider V_η as an operator $\mathcal{H}_\pi \to \mathrm{L}^2(G)$, with the obvious domain $\mathrm{dom}(V_\eta) = \{\varphi \in \mathcal{H}_\pi : V_\eta \varphi \in \mathrm{L}^2(G)\}$. We call η **admissible** whenever $V_\eta : \mathcal{H} \to \mathrm{L}^2(G)$ is an isometric embedding, and in this case V_η is called **(generalized) wavelet transform**. While the definition itself is rather simple, the problem of identifying admissible vectors is highly nontrivial, and the question whether these vectors exist for a given representation does not have a simple general answer. It is the main purpose of this book to develop in a systematical fashion criteria to deal with both problems.

As pointed out in [60], the construction principle for wavelet transforms had also been studied in mathematical physics, where admissible vectors η are called **fiducial vectors**, systems of the type $\{\pi(x)\eta : x \in G\}$ **coherent state systems**, and the corresponding inversion formulae **resolutions of the identity**; see [1, 73] for more details and references.

Here the earliest and most prominent examples were the original coherent states obtained by time-frequency shifts of the Gaussian, which were studied in quantum optics [114]. Perelomov [97] discussed the existence of resolutions of the identity in more generality, restricting attention to irreducible representations of unimodular groups. In this setting **discrete series** representations, i.e., irreducible subrepresentations of the regular representation λ_G of G turned out to be the right choice. Here every nonzero vector is admissible up to normalization. Moreover, Perelomov devised a construction which gives rise to resolutions of the identity for a large class of irreducible representations which were not in the discrete series. The idea behind this construction was to replace the group as integration domain by a well-chosen quotient, i.e., to construct isometries $\mathcal{H}_\pi \hookrightarrow \mathrm{L}^2(G/H)$ for a suitable closed subgroup H. In all of these constructions, irreducibility was essential: Only the well-definedness and a suitable intertwining property needed to be proved, and Schur's lemma would provide for the isometry property.

While we already remarked that [60] was not the first source to comment on the role of the affine group in constructing inversion formulae, suitably general criteria for nonunimodular groups were missing up to this point. Grossmann, Morlet and Paul showed how to use the orthogonality relations, established for these groups by Duflo and Moore [38], for the characterization of admissible vectors. More precisely, Duflo and Moore proved the existence of a uniquely defined unbounded selfadjoint operator C_π associated to a discrete series representation such that a vector η is admissible iff it is contained in the domain of C_π, with $\|C_\pi \eta\| = 1$. A second look at the admissibility condition (1.1) shows that in the case of the wavelet transform on $\mathrm{L}^2(\mathbb{R})$ this operator is given on the Plancherel transform side by multiplication with $|\omega|^{-1/2}$. This framework allowed to construct analogous transforms in a variety of settings, which was to become an active area of research in the subsequent years; a by no means complete list of references is [93, 22, 25, 48, 68, 49, 50, 51, 83, 7, 8]. See also [1] and the references therein.

However, it soon became apparent that admissible vectors exist outside the discrete series setting. In 1992, Mallat and Zhong [92] constructed a transform related to the original continuous wavelet transform, called the **dyadic wavelet transform**. Starting from a function $\psi \in \mathrm{L}^2(\mathbb{R})$ satisfying the **dyadic admissibility condition**

$$\sum_{n \in \mathbb{Z}} |\widehat{\psi}(2^n \omega)|^2 = 1 \quad , \quad \text{for almost every } \omega \in \mathbb{R} \tag{1.3}$$

one obtains the (weak-sense) inversion formula

$$f = \int_{\mathbb{R}} \sum_{n \in \mathbb{Z}} \langle f, \psi_{b,2^n} \rangle \psi_{b,2^n} 2^{-n} db \quad , \tag{1.4}$$

or equivalently, an isometric **dyadic wavelet transform** $\mathrm{L}^2(\mathbb{R}) \to \mathrm{L}^2(\mathbb{R} \times \mathbb{Z}, db2^{-n}dn)$, where dn denotes counting measure. Clearly the representation behind this transform is just the restriction of the above representation π to the closed subgroup $H = \{(b, 2^n) : b \in \mathbb{R}, n \in \mathbb{Z}\}$ of G, and the measure underlying the dyadic inversion formula is the left Haar measure of that subgroup. However, in one respect the new transform is fundamentally different: The restriction of π to H is no longer irreducible, in fact, it does not even contain irreducible subrepresentations (see Example 2.36 for details). Therefore (1.3) and (1.4), for all the apparent similarity to (1.1) and (1.2), cannot be treated in the same discrete series framework.

The example by Mallat and Zhong, together with results due to Klauder, Isham and Streater [67, 74], was the starting point for the work presented in this book. In each of these papers, a more or less straightforward construction led to admissibility conditions – similar to (1.1) and (1.3) – for representations which could not be dealt with by means of the usual discrete series arguments. The initial motivation was to understand these examples under a representation-theoretic perspective, with a view to providing a general strategy for the systematic construction of wavelet transforms.

The book departs from a few basic realizations: Any wavelet transform V_η is a unitary equivalence between π and a subrepresentation of λ_G, the left regular representation of G on $\mathrm{L}^2(G)$. Hence, the **Plancherel decomposition** of the latter into a direct integral of irreducible representations should play a central role in the study of admissible vectors, as it allows to analyze invariant subspaces and intertwining operators.

A first hint towards direct integrals had been given by the representations in [67, 74], which were constructed as direct integrals of irreducible representations. However, the particular choice of the underlying measure was not motivated, and it was unclear to what extent these constructions and the associated admissibility conditions could be generalized to other groups. Properly read, the paper by Carey [29] on reproducing kernel subspaces of $L^2(G)$ can be seen as a first source discussing the role of Plancherel measure in this context.

1.2 Overview of the Book

The contents of the remaining chapters may be roughly summarized as follows:

2. Introduction to the group-theoretic approach to the construction of continuous wavelet transforms. Embedding the discussion into $L^2(G)$. Formulation of a list of tasks to be solved for general groups. Solution of these problems for the toy example $G = \mathbb{R}$.
3. Introduction to the Plancherel transform for type I groups, and to the necessary representation-theoretic machinery.
4. Plancherel inversion and admissibility conditions for type I groups. Existence and characterization of admissible vectors for this setting.
5. Examples of admissibility conditions in concrete settings, in particular for quasiregular representations.
6. Sampling theory on the Heisenberg group.

Chapter 2 is concerned with the collection of basic notions and results, concerning coefficient operators, inversion formulae and their relation to convolution and the regular representations. In this chapter we formulate the problems which we intend to address (with varying degrees of generality) in the subsequent chapters. We consider existence and characterization of inversion formulae, the associated reproducing kernel subspaces of $L^2(G)$ and their properties, and the connection to discretization of the continuous transforms and sampling theorems on the group. Support properties of the arising coefficient functions are also an issue. Section 2.7 is crucial for the following parts: It discusses the solution of the previously formulated list of problems for the special case $G = \mathbb{R}$. It turns out that the questions mostly translate to elementary problems in real Fourier analysis.

Chapter 3 provides the "Fourier transform side" for locally compact groups of type I. The Fourier transform of such groups is obtained by integrating functions against irreducible representations. The challenge for Plancherel theory is to construct from this a unitary operator from $L^2(G)$ onto a suitable direct integral space. This problem may be seen as analogous to the case of the reals, where the tasks consists in showing that the Fourier transform defined on $L^1(\mathbb{R})$ induces a unitary operator $L^2(\mathbb{R}) \to L^2(\mathbb{R})$. However, for arbitrary locally compact groups the right hand side first needs to be constructed, which involves a fair amount of technique. The exposition starts from a representation-theoretic discussion of the toy example, and during the exposition to follow we refer repeatedly to this initial example.

Chapter 4 contains a complete solution of the existence and characterization of admissible vectors, at least for type I groups and up to unitary equivalence. The technique is a suitable adaptation of the Fourier arguments used for the toy example. It relies on a pointwise Plancherel inversion formula, which in this generality has not been previously established. In the course of argument we derive new results concerning the Fourier algebra and Fourier inversion on type I locally compact groups, as well as an L^2-version of the convolution theorem, which allows a precise description of L^2-convolution operators, including domains, on the Plancherel transform side 4.18. We comment on an interpretation of the support properties obtained in Chapter 2 in connection with the so-called "qualitative uncertainty principle". Using existence and uniqueness properties of direct integral decompositions, we then describe a general procedure how to establish the existence and criteria for admissible vectors (Remark 4.30). We also show that these criteria in effect characterize the Plancherel measure, at least for unimodular groups. Section 4.5 shows how the Plancherel transform view allows a unified treatment of wavelet and Wigner transforms associated to nilpotent Lie groups.

Chapter 5 shows how to put the representation-theoretic machinery developed in the previous chapters to work on a much-studied class of concrete representations, thereby considerably generalizing the existing results and providing additional theoretic background. We discuss semidirect products of the type $\mathbb{R}^k \rtimes H$, with suitable matrix groups H. These constructions have received considerable attention in the past. However, the representation-theoretic results derived in the previous chapters allow to study generalizations, e.g. groups of the sort $N \rtimes H$, where N is a homogeneous Lie group and H is a one-parameter group of dilations on N. The discussion of the Zak-transform in the context of Weyl-Heisenberg frames gives further evidence for the scope of the general representation-theoretic approach.

The final chapter contains a discussion of sampling theorems on the Heisenberg group $\mathbb{H}$. We obtain a complete characterization of the closed leftinvariant subspaces of $\mathrm{L}^2(\mathbb{H})$ possessing a sampling expansion with respect to a lattice. Crucial tools for the proof of these results are provided by the theory of Weyl-Heisenberg frames.

1.3 Preliminaries

In this section we recall the basic notions of representation theory, as far as they are needed in the following chapter. For results from representation theory, the books by Folland [45] and Dixmier [35] will serve as standard references.

The most important standing assumptions are that *all locally compact groups in this book are assumed to be Hausdorff and second countable* and *all Hilbert spaces in this book are assumed to be separable.*

Hilbert Spaces and Operators

Given a Hilbert space $\mathcal{H}$, the space of bounded operators on it is denoted by $\mathcal{B}(\mathcal{H})$, and the operator norm by $\|\cdot\|_\infty$. $\mathcal{U}(\mathcal{H})$ denotes the group of unitary operators on $\mathcal{H}$. Besides the norm topology, there exist several topologies of interest on $\mathcal{B}(\mathcal{H})$. Here we mention the **strong operator topology** as the coarsest topology making all mappings of the form

$$\mathcal{B}(\mathcal{H}) \ni T \mapsto T\eta \in \mathcal{H} \quad ,$$

with $\eta \in \mathcal{H}$ arbitrary, continuous, and the **weak operator topology**, which is the coarsest topology for which all coefficient mappings

$$\mathcal{B}(\mathcal{H}) \ni T \mapsto \langle \varphi, T\eta \rangle \in \mathbb{C},$$

with $\varphi, \eta \in \mathcal{H}$ arbitrary, are continuous. Furthermore, let the **ultraweak topology** denote the coarsest topology for which all mappings

$$\mathcal{B}(\mathcal{H}) \ni T \mapsto \sum_{n\in\mathbb{N}} \langle \varphi_n, T\eta_n \rangle$$

are continuous. Here $(\eta_n)_{n\in\mathbb{N}}$ and $(\varphi_n)_{n\in\mathbb{N}}$ range over all families fulfilling

$$\sum_{n\in\mathbb{N}} \|\eta_n\|^2 < \infty \quad , \quad \sum_{n\in\mathbb{N}} \|\varphi_n\|^2 < \infty \quad .$$

We use the abbreviations **ONB** and **ONS** for orthonormal bases and orthonormal systems, respectively. $\dim(\mathcal{H})$ denotes the Hilbert space dimension, i.e., the cardinality of an arbitrary ONB of $\mathcal{H}$. Another abbreviation is the word **projection**, which in this book always refers to selfadjoint projection operators on a Hilbert space. For separable Hilbert spaces, the Hilbert space dimension is in $\mathbb{N} \cup \{\infty\}$, where the latter denotes the countably infinite cardinal. The standard index set of cardinality m (wherever needed) is $I_m = \{1, \ldots, m\}$, where $I_\infty = \mathbb{N}$, and the standard Hilbert space of dimension m is $\ell^2(I_m)$.

If $(\mathcal{H}_i)_{i\in I}$ is a family of Hilbert spaces, then $\bigoplus_{i\in I} \mathcal{H}_i$ is the space of vectors $(\varphi_i)_{i\in I}$ in the cartesian product fulfilling in addition

$$\|(\varphi_i)_{i\in I}\|^2 := \sum_{i\in I} \|\varphi_i\|^2 < \infty \quad .$$

The norm thus defined on $\bigoplus_{i\in I} \mathcal{H}_i$ is a Hilbert space norm, and $\bigoplus_{i\in I} \mathcal{H}_i$ is complete with respect to the norm. If the $\mathcal{H}_i$ are orthogonal subspaces of a common Hilbert space $\mathcal{H}$, $\bigoplus_{i\in I} \mathcal{H}_i$ is canonically identified with the closed subspace generated by the union of the $\mathcal{H}_i$.

If T is a densely defined operator on $\mathcal{H}$ which has a bounded extension, we denote the extension by $[T]$.

Unitary Representations

A **unitary, strongly continuous representation**, or simply **representation**, of a locally compact group G is a group homomorphism $\pi : G \to \mathcal{U}(\mathcal{H}_\pi)$ that is continuous, when the right hand side is endowed with the strong operator topology. Since weak and strong operator topology coincide on $\mathcal{U}(\mathcal{H}_\pi)$, the continuity requirement is equivalent to the condition that all coefficient functions of the type

$$G \ni x \mapsto \langle \varphi, \pi(x)\eta \rangle \in \mathbb{C},$$

are continuous.

Given representations σ, π, and operator $T : \mathcal{H}_\sigma \to \mathcal{H}_\pi$ is called **intertwining operator**, if $T\sigma(x) = \pi(x)T$ holds, for all $x \in G$. We write $\sigma \simeq \pi$ if σ and π are **unitarily equivalent**, which means that there is a *unitary* intertwining operator $U : \mathcal{H}_\sigma \to \mathcal{H}_\pi$. It is elementary to check that this defines an equivalence relation between representations. For any subset $\mathcal{K} \subset \mathcal{H}_\pi$ we let

$$\pi(G)\mathcal{K} = \{\pi(x)\eta : x \in G, \eta \in \mathcal{K}\} \quad .$$

A subspace of $\mathcal{K} \subset \mathcal{H}_\pi$ is called **invariant** if $\pi(G)\mathcal{K} \subset \mathcal{K}$. Orthogonal complements of invariant subspaces are invariant also. Restriction of a representation to invariant subspaces gives rise to **subrepresentations**. We write $\sigma < \pi$ if σ is unitarily equivalent to a subrepresentation of π. σ and π are called **disjoint** if there is no nonzero intertwining operator in either direction. A vector $\eta \in \mathcal{H}_\pi$ is called **cyclic** if $\pi(G)\eta$ spans a dense subspace of $\mathcal{H}_\pi$. A **cyclic representation** is a representation having a cyclic vector. All representations of interest to us are cyclic. In particular our standing assumption that G is second countable implies that all representations occurring in the book are realized on separable Hilbert spaces. π is called **irreducible** if every nonzero vector is cyclic, or equivalently, if the only closed invariant subspaces of $\mathcal{H}_\pi$ are $\{0\}$ and $\mathcal{H}_\pi$. Given a family $(\pi_i)_{i\in I}$, the direct sum $\pi = \bigoplus_{i\in I} \pi_i$ acts on $\bigoplus_{i\in I} \mathcal{H}_{\pi_i}$ via

$$\pi(x)\left(\varphi_i\right)_{i\in I} = \left(\pi_i(x)\varphi_i\right)_{i\in I} \quad .$$

The main result in connection with irreducible representations is Schur's lemma characterizing irreducibility in terms of intertwining operators. See [45, 3.5] for a proof.

Lemma 1.1. *If π_1, π_2 are irreducible representations, then the space of intertwining operators between π_1 and π_2 has dimension 1 or 0, depending on $\pi_1 \simeq \pi_2$ or not.*
In other words, π_1 and π_2 are either equivalent or disjoint.

Using the spectral theorem the following generalization can be shown. The proof can be found in [66, 1.2.15].

Lemma 1.2. *Let π_1, π_2 be representations of G, and let $T : \mathcal{H}_{\pi_1} \to \mathcal{H}_{\pi_2}$ be a closed intertwining operator, defined on a dense subspace $\mathcal{D} \subset \mathcal{H}_{\pi_1}$. Then $\overline{\mathrm{Im}T}$ and $(\ker T)^\perp$ are invariant subspaces and π_1, restricted to $(\ker T)^\perp$, is unitarily equivalent to the restriction of π_2 to $\overline{\mathrm{Im}T}$).*
If, moreover, π_1 is irreducible, T is a multiple of an isometry.

Given G, the **unitary dual** $\widehat{G}$ denotes the equivalence classes of irreducible representations of G. Whenever this is convenient, we assume the existence of a fixed choice of representatives of $\widehat{G}$, taking recourse to Schur's lemma to identify arbitrary irreducible representations with one of the representatives by means of the essentially unique intertwining operator.

We next describe the contragredient $\overline{\pi}$ of a representation π. For this purpose we define two involutions on $\mathcal{B}(\mathcal{H}_\pi)$, which are closely related to taking adjoints. For this purpose let $T \in \mathcal{B}(\mathcal{H}_\pi)$. If $(e_i)_{i\in I}$ is any orthonormal basis, we may define two linear operators T^t and $\overline{T}$ by prescribing

$$\langle T^t e_i, e_j\rangle = \langle Te_j, e_i\rangle \quad , \quad \langle \overline{T} e_i, e_j\rangle = \overline{\langle Te_i, e_j\rangle} \quad .$$

It is straightforward to check that these definitions do not depend on the choice of basis, and that $T^* = \overline{T}^t$, as we expect from finitedimensional matrix calculus. Additionally, the relations $\overline{T^t} = \overline{T}^t = T^*$ and $(ST)^t = T^t S^t$, $\overline{ST} = \overline{S}\,\overline{T}$ are easily verified.

Now, given a representation $(\pi, \mathcal{H}_\pi)$, the **(standard realization of the) contragredient representation** $\overline{\pi}$ acts on $\mathcal{H}_\pi$ by $\overline{\pi}(x) = \overline{\pi(x)}$. In general, $\overline{\pi} \not\simeq \pi$.

Commuting Algebras

The study of the commuting algebra, i.e., the bounded operators intertwining a representation with itself, is a central tool of representation theory. In this book, the **commutant** of a subset $M \subset \mathcal{B}(\mathcal{H})$, is denoted by M', and it is given by

$$M' = \{T \in \mathcal{B}(\mathcal{H}) : TS = ST \quad , \quad \forall S \in M\} \quad .$$

It is a **von Neumann algebra**, i.e. a subalgebra of $\mathcal{B}(\mathcal{H})$ which is closed under taking adjoints, contains the identity operator, and is closed with respect to the strong operator topology. The von Neumann density theorem [36, Theorem I.3.2, Corollary 1.3.1] states for selfadjoint subalgebras $\mathcal{A} \subset \mathcal{B}(\mathcal{H})$, that closedness in any of the above topologies on $\mathcal{B}(\mathcal{H})$ is equivalent to $\mathcal{A} = \mathcal{A}''$.

There are two von Neumann algebras associated to any representation π, the **commuting algebra** of π, which is the algebra $\pi(G)'$ of bounded operators intertwining π with itself, and the **bicommutant** $\pi(G)''$, which is the von Neumann algebra generated by $\pi(G)$. Since $\mathrm{span}(\pi(G))$ is a selfadjoint algebra, the von Neumann density theorem entails that it is dense in $\pi(G)''$ with respect to any of the above topologies. Invariant subspaces are conveniently discussed in terms of $\pi(G)'$, since a closed subspace $\mathcal{K}$ is invariant under π iff the projection onto $\mathcal{K}$ is contained in $\pi(G)'$.