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Edited by A. Dold and B. Eckmann

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A Constructive Study via the Theory of the I*-measure

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PREFACE

D. Sullivan has discovered the following remarkable fact: The differential graded algebra (abbreviated DGA) of differential forms on a differential manifold contains information not only on the infinite part of the cohomology ring of the manifold, but also on the infinite part of the homotopy ring under the Whitehead product. It was also D. Sullivan who extended the notion of DGA of differential forms to an arbitrary simplicial complex. Moreover, he introduced the notion of minimal model for such a DGA, and showed that the same relations to homology and homotopy hold as in the case of manifolds; these relations can easily be deduced from such a minimal model. Though the main ideas of the theory may be traced back, as shown by Sullivan himself, to many predecessors from E. Cartan onwards, it is Sullivan who is the uncontestable founder of this whole theory which immensely influences further developments of algebraic topology. The present book aims at presenting the theory in an elementary form under the name of I^*-measure which is a synonym of "rational homotopy type" or "minimal model" in current use. We adopt this name to give evidence of its measure-theoretical character, with particular emphasis on its calculability. In fact Sullivan has expressed the opinion that one of the advantages of this theory lies in offering "a new concreteness for calculation". We introduce here the notion of calculability of a certain measure with respect to a certain geometrical construction in a technical sense. It is one of the aims of this book to show that the I^*-measure is calculable with respect to most of the important geometrical constructions usually encountered in algebraic topology, in contrast to the fact that even the simplest classical cohomology H^*-measure is not calculable, even with respect to the simplest geometrical constructions such as the addition (i.e., union) and multiplication (i.e. product-formation of spaces). Quite often explicit formulae exhibiting such calculability will be given, which put in evidence the concreteness of calculation for the measure in question.

The book is divided into seven chapters. Chapter I, which is introductory, introduces the basic notions of measure and its calculability. Chapter II, of a purely algebraic character, is concerned with the notion of differential graded algebras over a field $\underline{k}$ of characteristic 0, to be abbreviated as DGA/$\underline{k}$ or simply DGA, with the homology of a DGA, as well as the homotopy of DGA-morphisms; and, what is of utmost importance, with the notion of minimal model of a DGA $\mathcal{A}$, denoted as min $\mathcal{A}$, and various properties connected therewith. Chapter III introduces the notion of the deRham-Sullivan measure $A^*(K)$, the DGA of differential forms on a simplicial complex K, and the fundamental notion of I^*-measure $I^*(K)$ of K, defined simply as the minimal model of $A^*(K)$: $I^*(K) = \min A^*(K)$. Moreover, in Chapter III the homotopy invariance of $I^*(K)$ is proved as well as the generalized deRham theorem due to Sullivan, viz.

$$H(I^*(K)) \approx H(A^*(K)) \approx H^*_{\underline{k}}(K) ,$$

which shows that $I^*(K)$ contains complete information on the cohomology of K with coefficients in a field $\underline{k}$ of characteristic 0. Chapter IV shows that the I^*-measure $I^*(K)$ also contains complete information about the homotopy ring of K und the Whitehead product after tensoring with the basic field $\underline{Q}$ of rationals. Thus Chapters III and IV together give the fundamentals of the theory of I^*-measure in showing its connection with the classical homology and homotopy measures of a space. Chapter V gives a concrete example of actually calculating the I^*-measure in the important case of a homogeneous space. It begins with the work of E. Cartan and ends with an extension of a theorem of H. Cartan on real cohomology to the case of an I^*-measure. The known published proofs of this theorem are quite complicated; the proof given here follows closely the paper of Rashevskü which seems to be somewhat simpler. Chapter VI deals with the calculability of I^*-measure with respect to addition (or union) of two complexes, which is not so evident, in contrast to the multiplication (or product-formation) of two complexes which is entirely trivial. In this Chapter we give some explicit formulae for the determination of the I^*-measure of the resulting complex in terms of those of the component complexes, as well as the interrelated morphisms. As a consequence we show how to calculate, at least theoretically, the I^*-measure of a finite complex in an algorithmic manner, once the combinatorial structure of the complex is known. In this way we also establish an axiomatic system for the I^*-measure of finite complexes in a manner quite different from that of the well-known Eilenberg-Steenrod axiomatic system for H^*-measures. As applications of the method developed in this Chapter we prove a theorem about the explicit determination of the I^*-measure of the fiber space in terms of those of the base and the fiber, for a fibration $F \subset E \to B$; namely,

$$I^*(E) \approx \min(I^*(B) \underset{\tau}{\otimes} I^*(F)) .$$

In taking homology of both sides, we then get

$$H^*_{\underline{k}}(E) \approx H(I^*(B) \underset{\tau}{\otimes} I^*(F)) ,$$

in which I^* cannot be replaced by the cohomology H^* unless very special conditions are imposed on the base or the fiber. This furnishes a concrete example of showing clearly the superiority of the I^*-measure over the classical H^*-measure in the case where the coefficient field $\underline{k}$ is of characteristic 0. In the formulae above $\underset{\tau}{\otimes}$ means that the differential in the tensor product is twisted. The way of arriving at those formulae also shows clearly how such twisted differentials occur. The twisted differential can be explicitly determined in many important cases described in this and the next chapter. In the final Chapter VII various spectral sequences connected with fibrations are studied; they lead to an explicit determination of the I^*-measure of a fibre-square-constructed space. As an application we show that for a fibration with a homogeneous-space fiber with the usual structure group, the I^*-measure of the fiber space will be completely determined by those of the base and the fiber, and by certain characteristic elements which, when passing to homology, are just characteristic classes of the fibration in the usual sense.

Mathematics should be constructive. Therefore, in the treatment chosen in this book we stick as much as possible to a constructive point of view. Thus we impose in Chapter I three requirements for a measure to be efficient which include, besides invariance and calculability some kind of finiteness. The spaces considered will be restricted accordingly, to those having the homotopy type of a countable connected simplicial complex in the weak topology, although all the concepts and results are valid for much more general spaces. In Chapter III the proof of the deRham-Sullivan theorem is based on the one given originally by Weil with appropriate modifications to remove all traces of non-constructive character, so that the final isomorphism in the theorem is explicit and constructive. For the same reason in Chapter IV we have used the Cartan-Serre tower of more or less constructive nature, instead of the Postnikov tower of mere existential character. Such constructiveness is quite evident in our treatment in Chapters V and VI, and even in part of Chapter VII.

A brief outline of the present book has appeared in The Chern Symposium 1979. Essential part of it constituted the contents of a course on algebraic topology given at Berkeley in the spring semester of 1981. The author would like to express his gratitudes to many mathematicians of the United States, France and West-Germany for their invitation to visit their respective universities or institutes. They are too numerous to be all mentioned here but among them the author would like to mention in particular Prof. S.S. Chern of UC at Berkeley, Prof. A. Weil and Prof. J. Milnor of IAS, Prof. Kuiper of IHES, and Prof. Hirzebruch of MPI. The author has benefited greatly and in many respects from these relatively long periods of stay. For example, it was during his stay at IHES that he had the possibility to make personal contact with Prof. D. Sullivan and that he had paid a visit to the University of Lille, a center of development of Sullivan's theory; there the author was given a collection of Lille Publications, which was very helpful to his later research. It was also during his stay at MPI that he made acquaintance with Prof. H.J. Baues who was kind enough to give him access to his personal collection of published or unpublished literature on rational homotopy theory. Of particular importance to the author was the

visit of Profs. W. Browder, J. Spencer and F. Peterson to China in the spring of 1973. They brought us a copy of the Tokyo Symposium on Manifolds, and after their return to the USA Prof. F. Peterson has sent us a xerox copy of the Italian lecture notes on Sullivan's theory by Friedlander, Griffiths and Morgan, at that time yet hardly known even to the western mathematical world. Thanks to these precious presents the author became aware of and attracted by this beautiful theory and started his own research in the field.

Most of the results in this book have appeared in some form in the works of D. Sullivan. However, they have been clarified, generalized, discovered and rediscovered, formulated and reformulated by many others in various different presentations. It is indeed difficult to ascertain who should be considered the real contributor of this or that result, so that we have left aside all such questions of priority. Furthermore, the theory is now well-developed and has many ramifications, extensions, and applications. The author is quite unable to touch on these. In fact, he had to limit himself to a presentation of the most fundamental part of the theory centered around his own studies from his own point of view. In particular, the author very much regrets not even being able to give a sketch of the important contributions of Prof. K.T. Chen who may be considered as a co-founder of the theory, in quite a different setting. The bibliography will also be restricted to those items which are strongly related to the present text. However, besides the Italian lecture notes cited above, the author would like to mention in particular the following works which have been of great help, both for his research and for his writing:

1. S. Halperin: Lectures on minimal models, Lille Publications (1978 and 1981).
2. D. Lehmann: Théorie homotopique des formes différentielles d'après Sullivan, Astérisque (1977).
3. P. Déligne, P. Griffiths, J. Morgan and D. Sullivan: The real homotopy theory of Kähler manifolds, Invent. math. 29(1975), 245-254.
4. D. Sullivan: Infinitesimal computations in topology, Publ. Math. 47(1977), 269-331.

Finally, the author would like to express his gratitude to Mrs. Hermona Rosinger for her excellent typing. The author is also very grateful to the staff of Springer-Verlag in admitting this work to their publication list. Last but not least, the author expresses his gratitude to his wife, Chen Pi-ho; without her assistance in so many respects the writing of this book would have been impossible.

CONTENTS

Chapter I

FUNDAMENTAL CONCEPTS. MEASURE AND CALCULABILITY

I.1 THE NOTION OF MEASURE

Definition 1: A geometrical category GEOM is just a collection of geometrical objects of a certain kind. A geometrical construction CONS, which permits to construct in a definite manner a new geometrical object 0 from a finite set of geometrical objects $0_1, \ldots, 0_n$, is said to be pertaining to the geometrical category GEOM, if the new object 0 belongs to GEOM whenever $0_1, \ldots, 0_n$ belong to GEOM.

In other words, CONS is pertaining to GEOM if GEOM is closed with respect to CONS.

Definition 2: An algebraic category ALG is just a collection of algebraic objects of a certain kind. An algebraic construction CONS, which permits to construct in a definite manner a new algebraic object 0 from a finite set of algebraic objects $0_1, \ldots, 0_n$, is said to be pertaining to the algebraic category ALG if the new object 0 belongs to ALG whenever $0_1, \ldots, 0_n$ all belong to ALG. In other words, CONS is pertaining to ALG if ALG is closed with respect to CONS.

Definition 3: Given a geometrical category GEOM and an algebraic category ALG a correspondence Meas which associates to any object 0 in GEOM a definite object in ALG will be said to be an ALG-measure on GEOM or simply a measure on GEOM. In notation:

Meas: GEOM $\rightsquigarrow$ ALG.

For the usual algebraic categories we may mention the following trivial ones with the ordinary addition and multiplication as pertaining algebraic constructions:

$\underline{Z}$ = Collection of all integers,

$\underline{R}$ = Collection of all real numbers,

$\underline{C}$ = Collection of all complex numbers,

$\underline{Q}$ = Collection of all rational numbers

$A[t_1, \ldots, t_n]$ = Collection of all polynomials in indeterminates $t_1, \ldots, t_n$ with coefficients in A, a given commutative ring, etc.

In the examples below the algebraic category will not be specified.

Ex. 1:

GEOM = Collection of all polygons (connected or disconnected) in a euclidean plane.

CONS = Rigid motion, union, or intersection, etc.

Meas = Area of a polygon.

Similarly for collection of polyhedra in a 3-space as GEOM and volume as Meas.

Ex. 2:

GEOM = Collection of all bounded point sets in a plane.

CONS = Rigid motion, union, intersection, or formation of derived sets, etc.

Meas = Content or Lebesgue measure of a point set.

Ex. 3:

GEOM = Collection of all polygonal knots in a euclidean 3-space.

CONS = Certain allowable deformations, connected sum or amalgamation of two knots into one, etc.

Meas = Genus, signature, Alexander polynomial, or π_1 of the complement of a knot, etc.

Ex. 4:

GEOM = Collection of all finite linear graphs.

CONS = Formation of dual graph in the case of planar graphs, amalgamation of two graphs into one, etc.

Meas = Incidence matrix, chromatic polynomial, thickness, etc.

Ex. 5:

GEOM = Collections of various kinds of topological spaces, in particular complexes, manifolds, etc.

CONS = Formation of union and product of spaces, formation of universal covering space, shrinking of a subspace to a point, surgery in case of manifold, etc.

Meas = Dimension, Euler characteristic, betti numbers, etc. whenever they are well-defined.

Besides these numerical measures we encounter in algebraic topology various kinds of measures in the form of much more complicated algebraic structures like groups, rings, differential graded algebras, etc. Thus, we have:

Singular chain or cochain groups;

Homology or cohomology groups or rings over certain coefficient groups or rings;

Homotopy-group in various dimensions;

K-ring and various generalized cohomology rings, etc.

To put in evidence their measure character we shall adopt the following notations somewhat deviated from the usual ones (X = space in the given geometrical category, G = commutative group, A = commutative ring):

$H_n^G(X)$ instead of $H_n(X,G)$;

$H_G^n(X)$ instead of $H^n(X,G)$;

$H_\oplus^G(X)$ instead of $H_*(X,G)$ as a group;

$H_G^\oplus(X)$ instead of $H^*(X,G)$ as a group;

$H_A^*(X)$ instead of $H^*(X,A)$ as a ring;

etc.

I.2 EXAMPLES IN MEASURES WITH APPLICATIONS

Ex. A. The Brouwer fixed-point theorem

Let D^n be the closed unit disc in the euclidean n-space and $f : D^n \to D^n$ any map of D^n into itself. The Brouwer fixed-point theorem asserts the existence of a fixed point $x \in D^n$, such that $f(x) = x$.

We shall prove it by means of certain appropriate measures on some appropriate geometrical categories as follows.

Suppose the contrary, that no such fixed point exists. Let the boundary sphere of D^n be $S = S^{n-1}$. For any point $x \in S$ and any real number $t \in [0,1]$ the points tx and $f(tx)$ are then different, and their joining line in the direction from $f(tx)$ to tx will meet S in a unique point, say $g_t(x)$. For $0 \leq t \leq 1$ the maps

$$g_t : S \to S$$

give then a homotopy between the constant map g_0 and the identity map g_1 .

Consider now a geometrical category GEOM of spaces containing $S = S^{n-1}$ as a particular one and a group-measure M on GEOM verifying the following properties, which may also be considered as axioms about the measure in question:

A1. Every map

$$h : X \to Y$$

of spaces X, Y in GEOM will induce a morphism

$$h_M : M(X) \to M(Y) .$$

A2. As in A1 for maps homotopic to each other

$$h \simeq h' : X \to Y$$

the morphisms induced will be identical:

$$h_M = h'_M : M(X) \to M(Y) .$$

A3. For the particular space S the measure $M(S) \neq 0$ (i.e. $M(S)$ is non-trivial in the corresponding algebraïc category).

A4. For a particular constant map $h : S \to S$ the induced morphism is

$$h_M = 0 \quad (\text{i.e. } h_M(M(S)) \text{ is trivial or } = 0).$$

A5. For the particular identity map $h : S \to S$ the induced morphism is

$$h_M = \text{Identity} .$$

It is clear, that there will result some contradiction from the properties A1 – A5, if we assume, that no fixed point of f exists. Consequently the Brouwer theorem is true, and the above reasoning furnishes a proof in assuming the existence of a measure M verifying the properties A1 – A5. For such measures we may take e.g. H^G_{n-1} or π_{n-1}. Even π_n or π_{n+1} may be used as such measures in case $n \geq 3$, since $\pi_n(S^{n-1}) \approx \underline{Z}$ or $\underline{Z}_2$ and $\pi_{n+1}(S^{n-1}) \approx \underline{Z}_2$ verify the required properties.

The measures above are covariant in character, in that the induced morphism

$$h_M : M(X) \to M(Y) \quad \text{of a map} \quad h : X \to Y$$

is in the same direction as h from X to Y. We may also consider measures contravariant in character, with morphism induced by a map $h : X \to Y$ now in reverse direction as h, being from Y to X. It will then be denoted as

$$h^M : M(Y) \to M(X) .$$

The same reasoning may be applied to prove the Brouwer fixed point theorem, in assuming the existence of such a contravariant measure, verifying properties analogous to A1 – A5 with, however, all directions of morphisms to be reversed. The usual cohomology-group H_G^{n-1} is then an example of such a measure.

We may even take the integer measure M with $M(S) = \underline{Z}$ and

$$h_M \ (\text{or } h^M) : M(S) \to M(S)$$

for $h : S \to S$ given by

$$h_M(m) \ (\text{or } h^M(m)) = d(h) \cdot m , \qquad m \in \underline{Z} ,$$

where $d(h)$ is some integer, depending on the map h, to be called the <u>degree</u> of h. With the geometrical category GEOM consisting of a single space S and properties A1 – A5 we get again a proof of the theorem, which is in essence the original one of Brouwer.

Ex. B. <u>Fundamental Theorem of Algebra</u>

Let C be the complex plane and

$$f : C \to C$$

the map given by $(z \in C)$

$$f(z) = z^n + a_1 z^{n-1} + \dots + a_n$$

with $a_i \in C$. We shall prove, that $f(z) = 0$ for some $z \in C$ by means of certain appropriate measures on some appropriate geometrical category.

For this purpose let us set $a = \mathrm{Max}(|a_i|, 1)$ and take some $R > na$. Set

$$S = S(R) = \{z \in C / |z| = R\},$$
$$D = D(R) = \{z \in C / |z| \leq R\}.$$

There is then a radial projection

$$\pi = \pi(R) : C - (0) \to S .$$

Suppose that the theorem is not true, so that $f(z) \neq 0$ for all $z \in C$. Then

$$\pi f : D \to S \qquad .$$

is well defined. For $0 \leq t \leq 1$ define a map

$$f_t : C \to C$$

by

$$f_t(z) = z^n + t(a_1 z^{n-1} + \dots + a_n)$$

with

$$f_1 = f .$$

For $0 \le t \le 1$ and $z \in S$ we have

$$|f_t(z)| \ge |z^n| - t(|a_1|\,|z|^{n-1} + \dots + |a_n|) \ge R^n - na\,R^{n-1} > 0 .$$

Hence

$$\pi f_t : S \to S$$

is well-defined for any $0 \le t \le 1$ and defines a homotopy

$$\pi f \simeq g : S \to S$$

where g is given by

$$g(z) = z^n / R^{n-1} .$$

On the other hand, the map

$$\pi f'_t : S \to S$$

given by

$$\pi f'_t(z) = \pi f(tz)$$

for $z \in S$ is also well-defined for any $0 \le t \le 1$ and defines a homotopy

$$\pi f \simeq g' : S \to S$$

where

$$g'(z) = \pi f(0)$$
$$= \text{a constant} .$$

Suppose now, there is some geometrical category GEOM of spaces containing the circles S as particular ones, and a certain non-trivial group-measure M on GEOM , verifying properties B1 – B5 below:

B1 – B4. Same as A1 – A4 in Ex. A.