Lecture Notes in Mathematics 1552

Joachim Hilgert Karl-Hermann Neeb

Lie Semigroups and their Applications

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- an informative introduction, perhaps with some historical remarks: it should be accessible to a reader not particularly familiar with the topic treated;
- a subject index: as a rule this is genuinely helpful for the reader.

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Introduction

Although semigroups of transformations appear already in the original work of S. Lie as part of his efforts to find the right analogue of the theory of substitutions in the context of differential equations, it was Ch. Loewner who first studied such objects purposely [Loe88]. He considered semigroups of self-maps of the unit disc as a tool in geometric function theory. In the late seventies, subsemigroups of Lie groups were considered in relation to control systems with symmetries (cf. [JK81a,b], [Su72]). At about the same time Ol'shanskiĭ introduced such semigroups to in order to study the representation theory of infinite dimensional classical groups ([Ols91]). Moreover causality questions led people in relativity theory to consider subsemigroups of Lie groups generated by one-parameter semigroups as well. Motivated by this evidence Hofmann and Lawson worked out, in [HoLa83], systematic groundwork for a Lie theory of semigroups. These efforts eventually resulted in the monograph [HHL89].

In the meantime it has become increasingly clear that certain subsemigroups of Lie groups play a vital role in the harmonic analysis of symmetric spaces and representation theory. The purpose of this book is to lead the reader up to these applications of Lie semigroup theory. It is intended for a reader familiar with basic Lie theory but not having any experience with semigroups. In order to keep the overlap with [HHL89] to a minimum we have occasionally quoted theorems without proof from this book – especially when the version there is still the best available. On the other hand the last few years have seen rapid development, and so we are able to present improved versions of many results from [HHL89] with completely new proofs.

This book is not meant to be comprehensive. We have left out various topics that belong to the theory but, at the time being, don't show close connections with the applications we have in mind. Also we have chosen to focus on closed subsemigroups of Lie groups and thereby avoid certain technical complications.

A Lie semigroup is a closed subsemigroup S of a Lie group G which, as a closed subsemigroup, is generated by the images of all the one-parameter semigroups

$$\gamma_X : \mathbb{R}^+ \to S, \qquad t \mapsto \exp(tX).$$

The set of all these one-parameter semigroups can be viewed as a set $\mathbf{L}(S)$ in the Lie algebra $\mathfrak{g}$ of G. It is a closed convex cone satisfying

$$e^{\operatorname{ad} X}\, \mathbf{L}(S) = \mathbf{L}(S) \qquad \forall X \in \mathbf{L}(S) \cap -\mathbf{L}(S),$$

an algebraic identity which reflects the fact that S is invariant under conjugation by elements from the unit group $S \cap S^{-1}$. Convex cones satisfying these properties are called Lie wedges and play the role of Lie algebras in the Lie theory of semigroups. Following the general scheme of Lie theory one wants to study the properties of Lie semigroups via their Lie wedges using the exponential function for the translation mechanism. In Chapter 1 we describe the essential features of this mechanism. In particular the topological and algebraic obstructions that arise when one tries to find a Lie semigroup with prescribed Lie wedge are pointed out. The problem, called the globality problem, has not been solved in a definitive way, but one has far-reaching results which essentially reduce the globality problem to finding the maximal subsemigroups of Lie groups. The topic of maximal subsemigroups is taken up again later in the book (Chapter 6 and Chapter 8).

The main result of Chapter 1 is Theorem 1.35 which characterizes the Lie wedges that occur as the tangent wedge of a Lie semigroup in terms of the existence of certain functions on the group G. In order to prove it we use a result about ordered homogeneous spaces which is presented only later, in Chapter 4 (Cor. 4.22). We chose this way of organizing things to be able to present the globality problem without too much technical ballast. Moreover the material about ordered homogeneous spaces presented in Chapter 4 is of independent interest even though the separation between semigroup and ordered space aspects may seem artificial to insiders.

Chapter 2 is completely devoted to a list of examples which either have model character or serve as counterexamples at some point. In Chapter 3 we present various geometric and topological properties. Most importantly, it is shown that the interior of a Lie semigroup S is dense if $\mathbf{L}(S)$ generates $\mathfrak{g}$ as a Lie algebra (cf. Theorem 3.8). Also important for later applications is the fact that Lie semigroups admit simply connected covering semigroups (cf. Theorem 3.14).

In Chapter 5 some more consequences of the theory of ordered homogeneous spaces are listed. Among other things it is shown that the unit group of a Lie semigroup is connected. Moreover we explain how the existence of a Lie semigroup with prescribed Lie wedge in a given connected Lie group G is related to the existence of such a Lie semigroup in a covering group of G.

Chapter 6 deals with the characterization of maximal subsemigroups with interior points in simply connected groups with cocompact radical. They all have half-spaces as tangent wedges and a closed subgroup of codimension one as unit group. Finally we show how one can use this result to solve some controllability questions on reductive groups.

The main result of Chapter 7 is Lawson's Theorem on Ol'shanskiĭ semigroups which in particular says that for a connected Lie group G sitting inside a complexification $G_{\mathbb{C}}$ with a Lie algebra $\mathfrak{g}$ which admits a pointed $\mathrm{Ad}(G)$-invariant cone W with interior points, then $(g, X) \to g \exp iX$ is a homeomorphism $G \times W \to G \exp iW$ onto a closed subsemigroup of $G_{\mathbb{C}}$. This semigroup is called a complex Ol'shanskiĭ semigroup. Before we get there we show what consequences the existence of invariant cones with interior points has for a Lie algebra $\mathfrak{g}$, give a characterization of those Lie algebras containing pointed generating invariant cones, and describe the complete classification of such cones.

Complex Ol'shanskiĭ semigroups and their real analogues appear in many different contexts. They consist of the elements of $G_{\mathbb{C}}$ which map the positive part of the ordered homogeneous space $G_{\mathbb{C}}/G$ into itself, where the ordering is induced by the invariant cone field associated to the invariant cone. In the semisimple case they (at least the ones coming with the maximal invariant cones) can also be viewed as semigroups of compressions

$$\mathrm{compr}(\mathcal{O}) = \{g \in G_{\mathbb{C}} : g.\mathcal{O} \subseteq \mathcal{O}\}$$

of certain open G-orbits $\mathcal{O}$ in suitable flag manifolds associated with $G_{\mathbb{C}}$. In order to show this we study the open G-orbits on complex flag manifolds via the symplectic (in fact, pseudo-Kähler) structure that is given on these orbits. Using the results obtained in this process one can eventually show that complex Ol'shanskiĭ semigroups for maximal invariant cones are maximal subsemigroups.

Apart from their different geometric realizations, complex Ol'shanskiĭ semigroups occur as the natural domains in $G_{\mathbb{C}}$ to which one can analytically continue highest weight representations of G. We show in Chapter 9 how this is done for general G. Moreover we give some examples of this continuation procedure such as the holomorphic discrete series representations and the metaplectic representation which gives rise to Howe's oscillator semigroup. The largest subrepresentation of $L^2(G)$ - for general G - which admits an analytic continuation to a complex Ol'shanskiĭ semigroup leads to a Hardy space of holomorphic functions on this semigroup satisfying an L^2-condition on G-cosets. This Hardy space coincides with the classical notions for tube domains and polydiscs if G is a vector group or a torus.

In Chapter 10 we collect the results presented in this book for semigroups related to Sl(2).

For the orientation of the reader we conclude this introduction with some comments on the overlap with [HHL89].

Apart from some elementary properties of Lie wedges and cones Chaper 1 is independent of [HHL89]. The idea of monotone functions is only briefly touched in [HHL89] and the corresponding results we present in Chapter 1 are stronger and the proofs less complicated.

Some of the characteristic examples such as 2.1, parts of 2.2, 2.9 – 2.11 described in Chapter 2 occur also in [HHL89]. We have included them for the convenience of the reader since they illuminate some specific features of the theory.

Chapter 3 is independent of [HHL89]. The results of Section 3.2 were known at that time, but the new proof of Hofmann and Ruppert is shorter and it offers some new insights.

Ordered homogeneous spaces do also occur in [HHL89], where they are used to obtain the results about the structure of Lie semigroups near their group of units (cf. Sections 4.2, 4.3). The results on the global structure of ordered homogeneous spaces concerning properties such as global hyperbolicity are new (cf. Sections 4.4, 4.7).

The *Unit Group Theorem* and the *Unit Neighborhood Theorem* (cf. Section 5.1) were already proved in [HHL89]. Here we obtain these results out of a general theory of ordered homogeneous spaces.

Sections 6.2 – 6.6 are more or less contained in [HHL89]. The results in Sections 6.1 and 6.7 are new and complement the existing results in an interesting way. Since the area of maximal subsemigroups, in particular of maximal subsemigroups in simple groups, still presents many open problems, we decided to include the whole state of the theory of maximal semigroups in Chapter 6. We note also that the Sections 8.1 and 8.6 contain recent results on maximal subsemigroups in semisimple Lie groups complementing the material in Chapter 6 which is mostly concerned with groups G, where $\mathbf{L}(G)$ contains a compact Levi algebra.

Even though the theory of invariant cones and their classification by intersections with compactly embedded Cartan algebras is contained in [HHL89], our

Section 7.1 does not significantly overlap with [HHL89]. Our approach to invariant cones is based on coadjoint orbits. It seems to be more fruitful and far-reaching than the direct approach. Those results on invariant cones which we need in the sequel are proved along these lines. This made it possible to shorten some of the proofs considerably.

The remainder of Sections 7 – 9 is absolutely independent of [HHL89]. For more recent results lying already beyond the scope of this book and concerning the material contained in these sections we refer the reader to [Ne93a-f].

User's Guide

Since many results in this book do not depend on every preceding chapter, we give a list containing for each section, the set of all other sections on which it depends. If, e.g., Section 4.5 depends on Section 4.4 and Section 4.4 depends on Section 4.3, then Section 4.3 appears only in the list of Section 4.4. So the reader has to trace back the whole tree of references by using the lists of several sections. Nevertheless we hope that this is helpful to those readers interested merely in some specific sections of the book.

Chapter 1:

1.2 [1.1], 1.3 [1.1], 1.4 [1.3], 1.8 [1.7], 1.9 [1.1, 1.8], 1.10 [1.4, 1.9]

Chapter 2:

2.3 [1.7], 2.6 [1.4, 1.5, 1.9], 2.7 [1.10]

Chapter 3:

3.1 [1.5, 1.10, 2.2, 2.7], 3.3 [1.10, 2.11], 3.4 [1.10, 3.2], 3.5 [3.4], 3.6 [3.5]

Chapter 4:

4.2 [1.9, 4.1], 4.3 [1.4, 1.9, 4.1, 4.2], 4.4 [1.7, 4.3], 4.5 [4.3], 4.6 [3.2, 4.3], 4.7 [4.4, 4.6]

Chapter 5:

5.1 [1.4, 1.8, 4.3], 5.2 [1.10, 3.2, 4.3, 5.1], 5.3 [3.2, 4.4, 5.1], 5.4 [3.1, 4.2, 5.3], 5.5 [2.5, 5.4]

Chapter 6:

6.2 [2.9], 6.3 [1.7, 6.2], 6.4 [6.3], 6.5 [6.3], 6.6 [6.5], 6.7 [1.2, 1.10, 3.2, 4.2, 4.3, 6.6]

Chapter 7:

7.1 [1.2], 7.2 [1.3, 7.1], 7.3 [4.2, 5.3, 7.2]

Chapter 8:

8.1 [1.7], 8.4 [1.3, 1.7, 7.2, 8.1, 8.3], 8.5 [2.6, 7.2], 8.6 [6.7, 8.4, 8.5]

Chapter 9:

9.3 [3.4, 7.3], 9.4 [8.4], 9.5 [7.1], 9.7 [7.3]

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1. Lie semigroups and their tangent wedges

The basic feature of Lie theory is that of using the group structure to translate global geometric and analytic problems into local and infinitesimal ones. These questions are solved by Lie algebra techniques which are essentially linear algebra and then translated back into an answer to the original problem. Surprisingly enough it is possible to follow this strategy to a large extend also for semigroups, but things become more intricate. Because of the missing inverses one has not only to deal with linear algebra but also with convex geometry at the infinitesimal level. Similarly to the group case the Lie algebraic counterpart of a subsemigroup can either be defined as geometric (sub-)tangent vectors to the semigroup or a family of one-parameter semigroups contained in the semigroup (or at least in its closure). It turns out to be a convex cone in the Lie algebra, possibly containing non-trivial vector subspaces. For this reason we prefer the notion wedge. It is well known from Lie group theory that Lie subalgebras always correspond to analytic subgroups, but these analytic subgroups need not be closed, i.e., embedded manifolds. This difficulty of course does not disappear in the context of semigroups. But in order to avoid undue technical complications we will often simply restrict our attention to closed subsemigroups of Lie groups. Apart from the additional problems on the infinitesimal level caused by the replacement of vector spaces by wedges there is a new, even more serious, obstacle to a successful translation mechanism in the semigroup context: It is much harder to translate answers back to the global level since it turns out that the relation between Lie wedges and semigroups is quite complicated. In this chapter we study a class of semigroups which *can* be recovered from their Lie wedges, but before we do that, we collect in Section 1.1 a few facts about the geometry of wedges.

In Section 1.2 we consider wedges in vector space V which are invariant under the linear action of a compact group K. Of particular interest in this setting is the projection mapping which maps V onto the submodule of K-fixed points. These results will be used in Chapter 7 for invariant cones in Lie algebras and their duals.

Section 1.3 is a self-contained introduction into the characteristic function of a cone and its basic properties. This function is a basic tool in the study of groups acting on cones as automorphisms. It proves particularly valuable for non-compact groups.

In Section 1.4 we develop the notion of a Lie semigroup which will be fundamental throughout this book. To illustrate the condition that a Lie semigroup is determined by its infinitesimal data, a Lie wedge in the corresponding Lie algebra, we deal in some detail with the examples of compression semigroups of Lorentzian cones and Euclidean balls. As already mentioned above, the relation between Lie semigroups and Lie wedges incorporates some new difficulties which do not arise for Lie algebras and groups. These diffculties are described in a categorial framework in Sections 1.5 and 1.6.

In the remainder of Chapter 1 we are dealing with the globality problem, i.e., the problem to find for a given Lie wedge a Lie semigroup. The concept of a monotone function is a basic tool to cope with the globality problem. It is developed in Section 1.7 and some deeper insight into the existence of smooth and analytic monotone functions are gained in Section 1.8. The characterization of global Lie wedges by the existence of certain monotone functions is given in Section 1.9 but we postpone a technical part of the proof to Chapter 4, where we will use the theory of ordered homogeneous spaces to complete it. The last section contains various criteria for the globality of a Lie wedge which are deduced from the characterization from above and which will be useful throughout the other chapters.

1.1. Geometry of wedges

Let L be a finite dimensional vector space. A subset W is called a *wedge* if it is a closed convex cone. The vector space $H(W) := W \cap -W$ is called the *edge of the wedge*. We say that W is *pointed* if the edge of W is trivial and that W is *generating* if $W - W = L$. We denote the dual of L with L^*. The *dual wedge* $W^\star \subseteq L^*$ is the set of all functionals which are non-negative on W. Furthermore we set $\operatorname{algint} W := \operatorname{int}_{W-W} W$ and $W^\perp := H(W^\star)$. The following proposition is a collection of elementary facts about the relations between wedges and their duals.

Proposition 1.1. *We identify the dual of L^* with L. Then the following assertions hold for a wedge $W \subseteq L$:*

(i) $(W^\star)^\star = W$.

(ii) *W is generating if and only if $W^\star$ is pointed and, conversely, W is pointed if and only if $W^\star$ is generating.*

(iii) $\omega \in \operatorname{algint} W^\star$ *if and only if* $\omega(x) > 0$ *for all* $x \in W \setminus H(W)$ *and*

$$\operatorname{algint} W = \{x \in W : \omega(x) > 0 \ \textit{ for all } \ \omega \in W^\star \setminus H(W^\star)\}.$$

(iv) *For a family $(W_i)_{i\in I}$ of wedges in L we have that*

$$\Big(\bigcap_{i\in I} W_i\Big)^\star = \overline{\sum_{i\in I} W_i^\star} \quad \textit{and} \quad \Big(\sum_{i\in I} W_i\Big)^\star = \bigcap_{i\in I} W_i^\star.$$

(v) *If $V \subseteq L$ is a convex cone, then* $\overline{V} = (V^\star)^\star$, $\operatorname{algint} V = \operatorname{algint} \overline{V}$, *and* $\overline{V} = \overline{\operatorname{algint} V}$.

Proof. (i) It is clear that $W \subseteq (W^\star)^\star$. Let $x \notin W$. Then, by the Theorem of Hahn-Banach, there exists a $\omega \in W^\star$ with $\omega(x) < 0$. Therefore $x \notin (W^\star)^\star$.
(ii) Note first that

$$H(W^\star) = W^\perp = (W - W)^\perp.$$

Therefore $W^\star$ is pointed if and only if $W - W = L$, i.e., if W is generating. The dual assertion follows from (i) by applying the first one to $W^\star$.
(iii) In view of (i), it suffices to prove that

$$\operatorname{algint} W = \{x \in W : \omega(x) > 0 \text{ for all } \omega \in W^\star \setminus H(W^\star)\}.$$

"$\subseteq$": Let $x \in \operatorname{algint} W$. Then $0 \in \operatorname{int}_{W-W}(W - x)$, so that $W - W = W - \mathbb{R}^+ x$. If $\omega \in W^\star$ and $\omega(x) = 0$, then

$$\omega(W - W) = \omega(W) \subseteq \mathbb{R}^+.$$

Therefore $\omega(W - W) = 0$ since ω is linear, and hence $\omega \in H(W^\star)$.
"$\supseteq$": Suppose that $x \notin \operatorname{algint} W$. Then the Theorem of Hahn-Banach implies the existence of a linear functional $\omega' \neq 0$ on $W - W$ such that $\omega'(W) \subseteq \mathbb{R}^+$ and $\omega'(x) = 0$. Extending ω' to $\omega \in L^*$ we find that $\omega \in W^\star \setminus H(W^\star)$. This proves the inclusion $\supseteq$.
(iv) A linear functional is non-negative on $\sum_{i \in I} W_i$ if and only if it is non-negative on every wedge W_i, thus

$$(\sum_{i \in I} W_i)^\star = \bigcap_{i \in I} W_i^\star.$$

Replacing each wedge in this identity by its dual it follows with (i) that

$$(\sum_{i \in I} W_i^\star)^\star = \bigcap_{i \in I} W_i.$$

So the assertions follow from (i) and (v).
(v) Since every linear functional on L is continuous, it is clear that $V^\star = (\overline{V})^\star$. Therefore $(V^\star)^\star = \overline{V}$ follows from (i). To prove the rest of (v), in view of $\overline{V} \subseteq V - V$, we may assume that V is generating. First we note that $V + \operatorname{int} V \subseteq \operatorname{int} V$.

Let $x \in \operatorname{int} V$ and $v \in V$. Then $v + tx \in \operatorname{int} V$ for all $t > 0$ and therefore $v \in \overline{\operatorname{int} V}$. To see that $\operatorname{int} \overline{V} \subseteq \operatorname{int} V$ (the other inclusion is trivial), let $x \in \operatorname{int} \overline{V}$ and U a 0-neighborhood with $x - U \subseteq \overline{V}$. Set $W := U \cap \operatorname{int} V$. Then $x - W$ is an open subset of $\overline{V}$ and therefore it contains an element of $v \in V$. Then $v = x - w$ holds with $w \in W$, so $x = v + w \in V + \operatorname{int} V \subseteq \operatorname{int} V$. ∎

The geometry of a closed convex set in a finite dimensional vector space is completely determined by the set of extremal points . But between extremal points and the whole set one has interesting sets, the faces, which share properties of convex sets *and* extremal points. We only give the definitions in the context of wedges: Let F, $W \subseteq L$ be wedges. Then we set

$$L_F(W) := \overline{W + F - F} \quad \text{and} \quad T_F(W) := H\big(L_F(W)\big) = L_F(W) \cap -L_F(W).$$

The fact that $W + F - F$ is convex and stable under multiplication with non-negative scalars shows that $L_F(W)$ is a wedge. Note that $L_F(W) = \overline{W - F}$ if $F \subseteq W$. We say that a wedge $F \subseteq W$ is an *exposed face* of W if

$$F = W \cap T_F(W)$$

and a *face* of W if its complement $W \setminus F$ is an ideal in the additive semigroup W. The geometric meaning of these concepts will be clarified by the following two propositions. We write $\mathcal{F}(W)$ for the set of faces of W and $\mathcal{F}_e(W)$ for the set of exposed faces of W.

The following proposition describes how the faces of W and its dual wedge are related.

Proposition 1.2. *The set $\mathcal{F}_e(W)$ is stable under arbitrary intersections and therefore a complete lattice with $H(W)$ as minimal and W as maximal element. Moreover, the following assertions hold.*

(i) *The mappings*

$$\mathrm{op}^\star : \mathcal{F}_e(W^\star) \to \mathcal{F}_e(W),\ E \mapsto W \cap E^\perp$$

and

$$\mathrm{op} : \mathcal{F}_e(W) \to \mathcal{F}_e(W^\star),\ F \mapsto W^\star \cap F^\perp$$

are order reversing bijections mapping a face to its "opposite" face in the dual cone. Moreover, for every subset $E \subseteq W^\star$ the set

$$\mathrm{op}^\star(E) := E^\perp \cap W$$

is an exposed face of W and for every exposed face there exists $\omega \in W^\star$ with $F = \ker \omega \cap W$.

(ii) *For a wedge $F \subseteq W$ we have that*

$$L_F(W)^\star = W^\star \cap F^\perp.$$

Proof. Let $(F_i)_{i\in I}$ be a family of exposed faces of W and $F := \bigcap_{i\in I} F_i$. Then F is a wedge. The relation

$$F \subseteq W \cap T_F(W) \subseteq W \cap T_{F_i}(W) = F_i \qquad \forall i \in I$$

shows that $F = W \cap T_F(W) \in \mathcal{F}_e(W)$. We conclude that every non-empty subset in $\mathcal{F}_e(W)$ has an infimum. Thus this partially ordered set is a complete lattice. That $H(W)$ and W are the minimal and maximal elements, follows directly from the definition.

(ii) This follows from the definition of $L_F(W)$ and from Proposition 1.1(iv).

(i) Let $F \in \mathcal{F}_e(W)$. Then, in view of Proposition 1.1(ii) and (iv), we have

$$F^\star = \big(W \cap T_F(W)\big)^\star = \overline{W^\star - \mathrm{op}(F)} = L_{\mathrm{op}(F)}(W^\star). \tag{1.1}$$

Therefore

$$T_{\mathrm{op}(F)}(W^\star) \cap W^\star = W^\star \cap F^\perp = \mathrm{op}(F)$$

shows that $\mathrm{op}(F) \in \mathcal{F}_e(W^\star)$. Equation (1.1) and (ii) imply that

$$\mathrm{op}^\star \circ \mathrm{op} = \mathrm{id}_{\mathcal{F}_e(W)}.$$

Replacing W by $W^\star$ we also find that

$$\mathrm{op} \circ \mathrm{op}^\star = \mathrm{id}_{\mathcal{F}_e(W^\star)}.$$

Let $E \subseteq W^\star$ and $F := T_E(W^\star) \cap W^\star$ be the exposed face generated by E. Then

$$\mathrm{op}^\star(E) = L_E(W^\star)^\star = L_F(W^\star)^\star = \mathrm{op}^\star(F)$$

shows that $\mathrm{op}^\star(E)$ is an exposed face of W. Finally, suppose that $F \in \mathcal{F}_e(W)$ and take $\omega \in \mathrm{algint}\,\mathrm{op}(F)$. Then, by Proposition 1.1(iii),

$$\ker \omega \cap W = \mathrm{op}(F)^\perp \cap W = \mathrm{op}^\star \circ \mathrm{op}(F) = F.$$

■