Lecture Notes in Mathematics

1980

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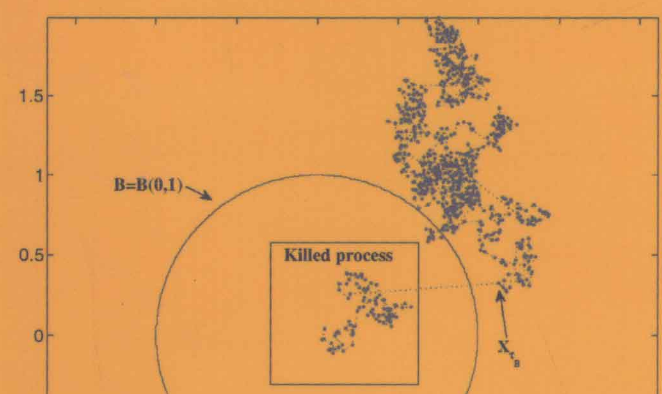
Potential Analysis of Stable Processes and its Extensions

Editors: Piotr Graczyk
Andrzej Stos



Springer

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ISBN: 978-3-642-02140-4 e-ISBN: 978-3-642-02141-1
DOI: 10.1007/978-3-642-02141-1

Lecture Notes in Mathematics ISSN print edition: 0075-8434
ISSN electronic edition: 1617-9692

Library of Congress Control Number: 2009928106

Mathematics Subject Classification (2000): 60J45, 60G52, 60J50, 60J75, 31B25, 31C05, 31C35, 31C25

Cover design: SPi Publisher Services

Printed on acid-free paper

springer.com

Foreword

This monograph is devoted to the potential theory of stable stochastic processes and related topics, such as the subordinate Brownian motions (including the relativistic process) and Feynman–Kac semigroups generated by certain Schrödinger operators.

The stable Lévy processes and related stochastic processes play an important role in stochastic modelling in applied sciences, in particular in financial mathematics, and the theoretical motivation for the study of their fine properties is also very strong. The potential theory of stable and related processes naturally extends the theory established in the classical case of the Brownian motion and the Laplace operator.

The foundations and general setting of probabilistic potential theory were given by G.A. Hunt [92](1957), R.M. Blumenthal and R.K. Getoor [23](1968), S.C. Port and J.C. Stone [130](1971). K.L. Chung and Z. Zhao [62](1995) have studied the potential theory of the Brownian motion and related Schrödinger operators. The present book focuses on classes of processes that contain the Brownian motion as a special case. A part of this volume may also be viewed as a probabilistic counterpart of the book of N.S. Landkof [117](1972).

The main part of Introduction that opens the book is a general presentation of fundamental objects of the potential theory of the isotropic stable Lévy processes in comparison with those of the Brownian motion (presented in a subsection). The introduction is accessible to a non-specialist. Also the chapters that follow should be of interest to a wider audience. A detailed description of the content of the book is given at the end of Chapter 1.

Some of the material of the book was presented by T. Byczkowski, T. Kulczycki, M. Ryznar and Z. Vondraček at the Workshop on Stochastic and Harmonic Analysis of Processes with Jumps held at Angers, France, May 2-9, 2006. The authors are grateful to the organizers and to the main supporters of the Workshop – the CNRS, the European Network of Harmonic Analysis HARP and the University of Angers – for this opportunity, which gave the incentive to write the monograph.

The book was written while Z. Vondraček was visiting the Department of Mathematics of University of Illinois at Urbana-Champaign. He thanks the department for the stimulating environment and hospitality. Thanks are also due to Andreas Kyprianou for several useful comments. The editors thank T. Luks for critical reading of some parts of the manuscript and for some of the figures illustrating the text.

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The research of this author is supported in part by MZOS grant 037-0372790-2801 of the Republic of Croatia.

Contents

Chapter 1
Introduction

1.1 Bases of Potential Theory of Stable Processes

In 1957, G. A. Hunt introduced and developed the potential theory of Markov processes in his fundamental treatise [92]. Hunt's theory is essentially based on the fact that the integral of the transition probability of a Markov process defines a **potential kernel**:

$$U(x, y) = \int_0^\infty p(t, x, y) dt\,.$$

One of the important topics in the theory is the study of multiplicative functionals of the Markov process, corresponding either to **Schrödinger perturbations** of the generator of the process, or to **killing the process** at certain stopping times. Among the most influential treatises on this subject are the monographs [23] by R. M. Blumenthal and R. K. Getoor, [60] by K. L. Chung, [22] by W. Hansen and J. Bliedtner, and [62] by K. L. Chung and Z. Zhao.

Harmonic functions of a strong Markov process are defined by the mean value property with respect to the distribution of the process stopped at the first exit time of a domain. An important case of such a function is the potential of a measure not charging the domain, thus yielding no "sources" to change the expected occupation time of the process.

To produce specific results, however, the general framework of Hunt's theory requires precise information on the asymptotics of the potential kernel of the given Markov process. For instance, the process of the Brownian motion in $\mathbb{R}^3$ is generated by the Laplacian, Δ, and yields the Newtonian kernel, $x \mapsto c|x-y|^{-1}$. Here y is the source or pole of the kernel. When x_0 is fixed and $|y| \to \infty$, we have that, regardless of x, $|x-y|^{-1}/|x_0-y|^{-1} \to 1$, which eventually leads to the conclusion that nonnegative functions harmonic on the whole of $\mathbb{R}^3$ must be constant.

Explicit formulas for the potential kernel are rare. Even the Brownian motion killed when first exiting a subdomain of $\mathbb{R}^d$ in general leads to a

K. Bogdan et al., *Potential Analysis of Stable Processes and its Extensions*, Lecture Notes in Mathematics 1980, DOI 10.1007/978-3-642-02141-1_1,

transition density and potential kernel which are not given by closed-form formulas, and may be even difficult to estimate.

A primary example of a jump process is the isotropic α-stable Lévy process in $\mathbb{R}^d$, whose potential kernel is the M. Riesz' kernel. The analytic theory of the Riesz kernel, the fractional Laplacian $\Delta^{\alpha/2}$, and the corresponding α-harmonic functions had been well established for a long time (see [133] and [117]). However, until recently little was known about the boundary behavior of α-harmonic functions on sub-domains of $\mathbb{R}^d$.

We begin the book by presenting some of the basic objects and results of the classical (Newtonian) potential theory ($\alpha = 2$), and Riesz potential theory ($0 < \alpha < 2$). We have already mentioned the well known but remarkable fact that the (Newtonian) potential theory of the Laplacian can be interpreted and developed by means of the Brownian motion ([71]). An analogous relationship holds for the (Riesz) potential theory of the fractional Laplacian and the isotropic α-stable Lévy process. We pursue this relationship in the following sections. We like to remark that $\Delta^{\alpha/2}$ is a primary example of a nonlocal pseudo differential operator ([97]) and we hope that a part of our discussion will extend to other nonlocal operators. Apart from its significance in mathematics, the fractional Laplacian appears in theoretical physics in the connection to the problem of stability of the matter [118]. Namely, the operator $I-(I-\Delta)^{1/2}$ corresponds to the kinetic energy of a relativistic particle and $\Delta^{1/2}$ can be regarded as an approximation to $I-(I-\Delta)^{1/2}$, see, e.g., [45], [134].

In what follows, functions and sets are assumed to be Borel measurable. We will write $f \approx g$ to indicate that f and g are *comparable*, i.e. there is a constant c (a positive real number independent of x), such that $c^{-1}f(x) \leqslant g(x) \leqslant cf(x)$. Values of *constants* may change from place to place, for instance $f(x) \leqslant (2c+1)g(x) = cg(x)$ should not alarm the reader.

1.1.1 Classical Potential Theory

We consider the Gaussian kernel,

$$g_t(x) = \frac{1}{(4\pi t)^{d/2}}\, e^{-|x|^2/4t}\,, \quad x \in \mathbb{R}^d\,, \quad t > 0\,. \tag{1.1}$$

It is well known that $\{g_t,\ t \geqslant 0\}$ form a convolution semigroup: $g_s * g_t = g_{s+t}$, where $s, t > 0$. This property is at the heart of the classical potential theory. Complicating the notation slightly, we define transition probability

$$g(s,x,t,A) = \int_{A-x} g_{t-s}(y)dy\,, \quad s < t\,,\ x \in \mathbb{R}^d\,,\ A \subset \mathbb{R}^d\,. \tag{1.2}$$

The semigroup property of $\{g_t\}$ is equivalent to the following Chapman-Kolmogorov equation

$$\int_{\mathbb{R}^d} g(s,x,u,dz)g(u,z,t,A) = g(s,x,t,A)\,, \quad s<u<t\,,\ x\in\mathbb{R}^d\,,\ A\subset\mathbb{R}^d\,.$$

If $d \geqslant 3$ then we define and calculate the Newtonian kernel,

$$N(x) = \int_0^\infty g_t(x)dt = \mathcal{A}_{d,2}|x|^{2-d}\,, \quad x\in\mathbb{R}^d\,.$$

Here and below

$$\mathcal{A}_{d,\gamma} = \Gamma((d-\gamma)/2)/(2^\gamma \pi^{d/2}|\Gamma(\gamma/2)|)\,. \tag{1.3}$$

The semigroup property yields that $N * g_s(x) = \int_s^\infty g_t(x)dt \leqslant N(x)$. Recall that a function $h\in C^2(D)$ is called *harmonic* in an open set $D\subseteq\mathbb{R}^d$ if it satisfies Laplace's equation,

$$\Delta h(x) = \sum_{i=1}^d \frac{\partial^2 h(x)}{\partial x_i^2} = 0\,, \quad x\in D\,. \tag{1.4}$$

It is well known that N is harmonic on $\mathbb{R}^d\setminus\{0\}$. Let $B(a,r)=\{x\in\mathbb{R}^d : |x-a|<r\}$, where $a\in\mathbb{R}^d$, $r>0$. We also let $B_r = B(0,r)$, $B=B_1=B(0,1)$. The *Poisson kernel* of $B(a,r)$ is

$$P(x,z) = \frac{\Gamma(d/2)}{2\pi^{d/2}r}\frac{r^2-|x-a|^2}{|x-z|^d}\,, \quad x\in B(a,r)\,, \quad z\in\partial B(a,r)\,. \tag{1.5}$$

It is well known that if h is harmonic in an open set containing the closure of $B(a,r)$ then

$$h(x) = \int_{\partial B(a,r)} h(z)\,P(x,z)\sigma(dz)\,, \quad x\in B(a,r)\,. \tag{1.6}$$

Here σ denotes the $(d-1)$-dimensional Haussdorff measure on $\partial B(a,r)$. We like to note that $P(x,z)$ is positive and continuous on $B(a,r)\times\partial B(a,r)$, and has the following properties:

$$\int_{\partial B(a,r)} P(x,z)\sigma(dz) = 1\,, \quad x\in B(a,r)\,, \tag{1.7}$$

$$\lim_{x\to w}\int_{\partial B(a,r)\setminus B(w,\delta)} P(x,z)\sigma(dz) = 0\,, \quad w\in\partial B(a,r)\,, \quad \delta>0\,. \tag{1.8}$$

It is also well known that for every $z \in \partial B(a,r)$, $P(\cdot, z)$ is harmonic in $B(a,r)$, a property resembling Chapman-Kolmogorov equation if we consider (1.6) for $h(x) = P(x, z_0)$. Consequently, if $f \in C(\partial B(a,r))$, then the Poisson integral,

$$P[f](x) = \int_{\partial B(a,r)} P(x,z)\, f(z)\sigma(dz)\,, \quad x \in B(a,r)\,, \tag{1.9}$$

solves the *Dirichlet problem* for $B(a,r)$ and f. Namely, $P[f]$ extends to the unique continuous function on $B(a,r) \cup \partial B(a,r)$, which is harmonic in $B(a,r)$, and coincides with f on $\partial B(a,r)$, see (1.8). In particular, $P[1] \equiv 1$, compare (1.7).

An analogous *Martin representation* is valid for every nonnegative h harmonic on $B(a,r)$,

$$h(x) = P[\mu](x) := \int_{\partial B(a,r)} P(x,z)\, \mu(dz)\,, \quad x \in B(a,r)\,. \tag{1.10}$$

Here $\mu \geqslant 0$ is a unique nonnegative measure on $\partial B(a,r)$. We like to note that appropriate *sections* of $P[\mu]$ *weakly* converge to μ ([107]), which reminds us that in general the boundary values of harmonic functions require handling with care.

By (1.5) we have that $P(x_1, z) \leqslant (1+s/r)^d(1-s/r)^{-d}P(x_2,z)$ if $x_1, x_2 \in B(a,s)$, $s < r$, $z \in \partial B(a,r)$. As a direct application of (1.10) we obtain the following *Harnack inequality*,

$$c^{-1}h(x_1) \leqslant h(x_2) \leqslant c\, h(x_1)\,, \quad x_1, x_2 \in B(a,s)\,, \tag{1.11}$$

provided h is *nonnegative* harmonic. We see that h is *nearly constant* (i.e. comparable with 1) on $B(a,s)$ for $s < r$. If D is *connected*, then considering finite coverings of *compact* $K \subset D$ by *overlapping chains* of balls, we see that nonnegative functions h harmonic on D are nearly constant on K, see Figure 1.1.

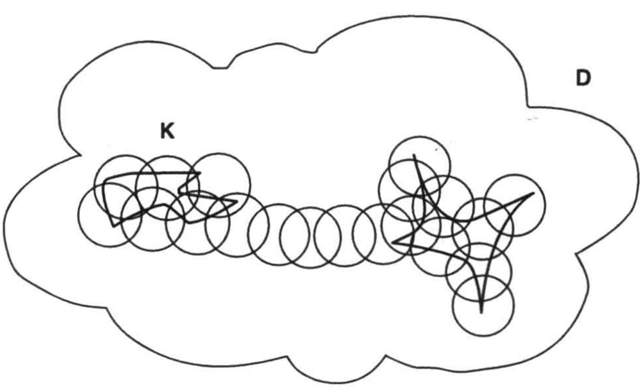

Fig. 1.1 Harnack chain

Despite its general importance, Harnack inequality is less useful at the boundary of the domain because the corresponding constant gets inflated for points close to the boundary. In fact, nonnegative harmonic functions present a complicated array of asymptotic behaviors at the boundary, see (1.5). To study the asymptotics, we first concentrate on nonnegative harmonic function *vanishing* at a part of the boundary.

The Boundary Harnack Principle (**BHP**) for *classical* harmonic functions delicately depends on the geometric regularity of the domain. To simplify our discussion we will consider the following Lipschitz condition. Let $d \geqslant 2$. Recall that $\Gamma : \mathbb{R}^{d-1} \to \mathbb{R}$ is called Lipschitz if there is $\lambda < \infty$ such that

$$|\Gamma(y) - \Gamma(z)| \leqslant \lambda |z - y|, \quad y, z \in \mathbb{R}^{d-1}. \tag{1.12}$$

We define (special Lipschitz domain)

$$D_\Gamma = \{x = (x_1, \dots, x_d) \in \mathbb{R}^d \,:\, x_d > \Gamma(x_1, \dots, x_{d-1})\}. \tag{1.13}$$

A nonempty open $D \subseteq \mathbb{R}^d$ is called a Lipschitz domain if for every $z \in \partial D$ there exist $r > 0$, a Lipschitz function $\Gamma : \mathbb{R}^{d-1} \to \mathbb{R}$, and an isometry T of $\mathbb{R}^d$, such that $D \cap B(z, r) = T(D_\Gamma) \cap B(z, r)$, that is, if D is locally isometric with a set "above" the graph of a Lipschitz function.

Theorem 1.1 (Boundary Harnack Principle). *Let D be a connected Lipschitz domain. Let $U \subset \mathbb{R}^d$ be open and let $K \subset U$ be compact. There exists $C < \infty$ such that for every (nonzero) functions $u, v \geqslant 0$, which are harmonic in D and vanish continuously on $D^c \cap U$, we have*

$$C^{-1}\frac{u(y)}{v(y)} \leqslant \frac{u(x)}{v(x)} \leqslant C\frac{u(y)}{v(y)}, \quad x, y \in K \cap D. \tag{1.14}$$

Thus, the ratio u/v is *nearly constant* on $D \cap K$. Furthermore, under the above assumptions,

$$\lim_{x \to z} \frac{u(x)}{v(x)} \quad \text{exists as } x \to z \in \partial D \cap K, \tag{1.15}$$

see Figure 1.2.

The theorem is crucial in the study of asymptotics and structure of general nonnegative harmonic functions in Lipschitz domains. The proof of **BHP** for classical harmonic functions in Lipschitz domains was independently given by B. Dahlberg(1977), A. Ancona(1978) and J.-M. Wu(1978), and (1.15) was published by D. Jerison and C. Kenig in 1982.

We now return to $\{g_t\}$, and the resulting transition probability g. By Wiener's theorem there are probability measures P^x, $x \in \mathbb{R}^d$, on the space of all *continuous* functions (paths) $[0, \infty) \ni t \mapsto X(t) \in \mathbb{R}^d$, such that $P^{x_0}(X(0) = x_0) = 1$ and $P^{x_0}(X_t \in A | X_s = x) = g(s, x, t, A) = P^x(X_{t-s} \in$

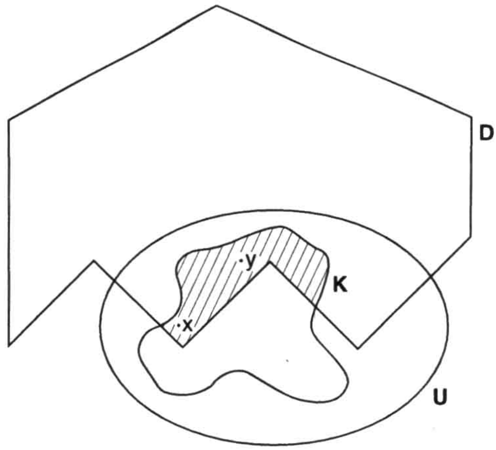

Fig. 1.2 The setup of **BHP**

A), for $x_0, x \in \mathbb{R}^d$, $0 \leqslant s < t$, $A \subset \mathbb{R}^d$. Recall that the construction of the distribution of the process from a transition probability on this path space requires certain continuity properties of the transition probability in time. Here we have $\lim_{t\to 0} g_t(x)/t = 0$ ($x \neq 0$), which eventually allows the paths to be continuous by Kolmogorov's test or by Kinney-Dynkin theorem, see [135]. Thus, X_t is continuous. Denote $X = (X_t) = (X_t^1, \ldots, X_t^d)$, and let E^x be the integration with respect to P^x. We have $E^0 X_t^i = 0$, $E^0 (X_t^i)^2 = 2t$. Thus, $X_t = B_{2t}$, where B_t is the usual Brownian motion with variance of each coordinate equal to t.

By the construction, $E^x f(X_t) = \int_{\mathbb{R}^d} f(y) g(0, x, t, dy) = \int_{\mathbb{R}^d} f(y) g_t(y - x) dy$ for $x \in \mathbb{R}^d$, $t > 0$, and nonnegative or integrable f. For a (Borel) set $A \subset \mathbb{R}^d$ by Fubini-Tonelli theorem,

$$E^x \int_0^\infty \mathbf{1}_A(X_t) dt = \int_A N(y - x) dy \,, \quad x \in \mathbb{R}^d \,.$$

Therefore $N(\cdot - x)$ may be interpreted as the density function of (the measure of) the expected occupation time of the process, when started at x.

So far we have only considered X evaluated at constant (deterministic) times t. For an open $D \subseteq \mathbb{R}^d$ we now define the *first exit time* from D,

$$\tau_D = \inf\{t > 0 : \, X_t \notin D\} \,.$$

By the usual convention, $\inf \emptyset = \infty$. τ_D is a Markov (stopping) time. A function h defined and Borel measurable on $\mathbb{R}^d$ is harmonic on D if for every open bounded set U such that $\overline{U} \subseteq D$ (denoted $U \subset\subset D$) we have

$$h(x) = E^x h(X_{\tau_U}) \,, \quad x \in U \,. \tag{1.16}$$

We assume here the absolute convergence of the integral. Since the P^x-distribution of $X_{\tau_{B(x,r)}}$ is the normalized surface measure on the sphere $\partial B(x,r)$, the equality (1.16) reads as follows:

$$h(x) = \int_{\partial B(x,r)} h(y)\, P(0, y - x)\sigma(dy)\,,$$

if $x \in U = B(a,r)$, see (1.5) and (1.6). Thus, (1.16) agrees with the classical definition of harmonicity.

The above definitions may and will be extended below to other strong Markov processes, and (1.16) may be referred to as the "averaging property" or "mean value property".

We should note that (for the Brownian motion) the values of h on D^c are irrelevant in (1.16) because $X_{\tau_U} \in \partial U \subset D$ in (1.16). For the isotropic stable Lévy process, which we will discuss below, the support of the distribution of the process stopped at the first exit time of a domain is typically *the whole complement* of the domain. Indeed, as time (t) advances, the paths of the process may leave the domain either by *continuously approaching* the boundary or by a direct *jump* to the complement of the domain. In particular, a harmonic function should generally be defined on the whole of $\mathbb{R}^d$. It is of considerable importance to classify nonnegative harmonic functions of the process according to these two scenarios, see the concluding remarks in [38].

To indicate the role of the strong Markov property, we consider a nonnegative function $\tilde{h}$ on D^c and we let $h(x) = E^x \tilde{h}(X_{\tau_D})$, $x \in D$. We will regard $\tilde{h}$ on D^c as the boundary/external values of h, as appropriate for general processes with jumps. It will be convenient to write $h(x)$ for $\tilde{h}(x)$ if $x \in D^c$. Let $x \in U \subset\subset D$. We have

$$E^x h(X_{\tau_U}) = E^x E^{X_{\tau_U}} h(X_{\tau_D}) = E^x h(X_{\tau_D}) = h(x)\,.$$

In particular we see that h is harmonic on U. The above essentially also proves that $\{h(X_{\tau_U})\}$ is a martingale ordered by the inclusion of (open relatively compact) subsets U of D, with respect to every P^x, $x \in D$. Closability of such martingales is of some interest in this theory [27, 38], and relates to the existence of boundary values of harmonic functions. For instance the martingales given by Poisson integrals (1.10) are not closable for singular measures μ on $\partial B(a,r)$.

1.1.2 Potential Theory of the Riesz Kernel

We will introduce the principal object of this book, namely the isotropic (rotation invariant) α-stable Lévy process. We will construct the transition density of the process by using convolution semigroups of measures. For a measure γ on $\mathbb{R}^d$, we let $|\gamma|$ denote its total mass. For a function f we let $\gamma(f) = \int f d\gamma$, whenever the integral makes sense. When $|\gamma| < \infty$ and $n = 1, 2, \ldots$ we let $\gamma^n = \gamma * \ldots * \gamma$ (n times) denote the n-fold convolution of γ with itself:

$$\gamma^n(f) = \int f(x_1 + x_2 + \cdots + x_n)\gamma(dx_1)\gamma(dx_2)\ldots\gamma(dx_n).$$

We also let $\gamma^0 = \delta_0$, the evaluation at 0. If γ is finite on $\mathbb{R}^d$ then we define

$$P_t^\gamma = \exp t(\gamma - |\gamma|\delta_0) := \sum_{n=0}^{\infty} \frac{t^n (\gamma - |\gamma|\delta_0)^n}{n!} \tag{1.17}$$

$$= (\exp -t|\gamma|\delta_0) * \exp t\gamma = e^{-t|\gamma|} \sum_{n=0}^{\infty} \frac{t^n \gamma^n}{n!}, \quad t \in \mathbb{R}. \tag{1.18}$$

By (1.18) each P_t^γ is a probability measure, provided $\gamma \geqslant 0$ and $t \geqslant 0$, which we will assume in what follows. By (1.17), P_t^γ form a convolution semigroup,

$$P_t^\gamma * P_s^\gamma = P_{s+t}^\gamma, \quad s, t \geqslant 0.$$

Furthermore, for two such measures γ_1, γ_2, we have

$$P_t^{\gamma_1} * P_t^{\gamma_2} = P_t^{\gamma_1+\gamma_2}, \quad t > 0.$$

By (1.17),

$$\lim_{t\to 0}(P_t^\gamma - \delta_0)/t = \gamma - |\gamma|\delta_0. \tag{1.19}$$

In the following discussion for simplicity we will also assume that γ has bounded support and that γ is symmetric: $\gamma(-A) = \gamma(A)$, $A \subset \mathbb{R}^d$. The reader may want to verify that

$$\int_{\mathbb{R}^d} |y|^2 P_t^\gamma(dy) = t \int_{\mathbb{R}^d} |y|^2 \gamma(dy) < \infty, \quad t \geqslant 0. \tag{1.20}$$

As a hint we note that only the third term in (1.17) contributes to (1.20). In particular,

$$P_t^\gamma(B(0,R)^c) \leqslant t \int_{\mathbb{R}^d} |y|^2 \gamma(dy)/R^2 \to 0 \quad \text{as } R \to \infty. \tag{1.21}$$

We define

$$\nu(B) = \mathcal{A}_{d,-\alpha} \int_B |z|^{-d-\alpha} dz, \quad B \subset \mathbb{R}^d. \tag{1.22}$$

It is a Lévy measure, i.e. a nonnegative measure on $\mathbb{R}^d \setminus \{0\}$ satisfying

$$\int_{\mathbb{R}^d} \min(|y|^2, 1)\, \nu(dy) < \infty \,. \tag{1.23}$$

We also note that ν is symmetric. We consider the following operator, the fractional Laplacian,

$$\Delta^{\alpha/2} u(x) = \mathcal{A}_{d,-\alpha} \lim_{\varepsilon \to 0+} \int_{\{y \in \mathbb{R}^d:\ |y-x|>\varepsilon\}} \frac{u(y)-u(x)}{|y-x|^{d+\alpha}} dy \,. \tag{1.24}$$

The limit exists if, say, u is C^2 near x and bounded on $\mathbb{R}^d$. The claim follows from Taylor expansion of u at x, with remainder of order two, and by the symmetry of ν. We like to note that $A = \Delta^{\alpha/2}$ satisfies the *positive maximum principle*: for every $\varphi \in C_c^\infty(\mathbb{R}^d)$

$$\sup_{y \in \mathbb{R}^d} \varphi(y) = \varphi(x) \geqslant 0 \quad \textit{implies} \quad A\varphi(x) \leqslant 0 \,.$$

The most general operators on $C_c^\infty(\mathbb{R}^d)$ which have this property are of the form

$$A\varphi(x) = \sum_{i,j=1}^{d} a_{ij}(x) D_{x_i} D_{x_j} \varphi(x) + b(x)\nabla\varphi(x) + q(x)\varphi(x) + \int_{\mathbb{R}^d} \left(\varphi(x+y) - \varphi(x) - y\nabla\varphi(x)\, \mathbf{1}_{|y|<1}\right) \mu(x, dy) \,. \tag{1.25}$$

Here $y\nabla\varphi$ is the scalar product of y and the gradient of φ, and for every x, $a(x) = (a_{ij}(x))_{i,j=1}^n$ is a real nonnegative definite symmetric matrix, the vector $b(x) = (b_i(x))_{i=1}^d$ has real coordinates, $q(x) \leqslant 0$, and $\mu(x, \cdot)$ is a Lévy measure. The description is due to Courrège, see [90, Proposition 2.10], [151, Chapter 2] or [97, Chapter 4.5]. For translation invariant operators of this type, a, b, q, and μ are independent of x. For $\Delta^{\alpha/2}$ we further have $a = 0$, $b = 0$, $q = 0$ and $\mu = \nu$.

For $r > 0$ and a function φ on $\mathbb{R}^d$ we consider its dilation $\varphi_r(y) = \varphi(y/r)$, and we note that $\nu(\varphi_r) = r^{-\alpha}\nu(\varphi)$. In particular, ν is homogeneous: $\nu(rB) = r^{-\alpha}\nu(B)$, $B \subset \mathbb{R}^d$. Similarly, if $\varphi \in C_c^\infty(\mathbb{R}^d)$, then $\Delta^{\alpha/2}(\varphi_r) = r^{-\alpha}(\Delta^{\alpha/2}\varphi)_r$.

We will consider approximations of ν and $\Delta^{\alpha/2}$ suggested by (1.24). For $0 < \delta \leqslant \varepsilon \leqslant \infty$ we define measures $\nu_{\delta,\varepsilon}(f) = \int_{\delta \le |y| < \varepsilon} f(y)\nu(dy)$. We have

$$P_t^{\nu_{\delta,\infty}} - P_t^{\nu_{\varepsilon,\infty}} = P_t^{\nu_{\varepsilon,\infty}} * \left(P_t^{\nu_{\delta,\varepsilon}} - \delta_0\right) \,. \tag{1.26}$$

When $\varepsilon \to 0$, the above converges (uniformly in δ) to 0 on each C_c^∞ function with compact support. This claim follows from Taylor expansion with the