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Fractional calculus and integral transforms of generalized functions

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Preface

In recent years there has been a lot of interest in extending the standard classical integral transforms to classes of generalised functions or distributions. The theory of the Fourier transform has been documented in standard works such as [24] and [75] but it was not until the appearance of [87] that transforms on the half-line $(0,\infty)$ such as the Laplace, Mellin and Hankel transforms received much attention. In the ten years since [87] was published, interest has continued. However, most authors have used classes of generalised functions which are ideal for the particular transform under consideration but for no others. The purpose of this book is to describe a class of spaces of generalised functions which are amenable to the study of a number of important operators and to use the theory to solve in some detail a number of standard problems. In particular, we show how various classical results are incorporated in our distributional theory.

Since an indefinite integral is probably the simplest integral transform of all, no apology is needed for using this as the starting point for a theory of fractional calculus, another topic which has sprung to life in recent years with the publication of [56] and [74]. This distributional fractional calculus is used as a unifying theme in the later chapters of the book. We have concentrated on problems which are of general interest and where the theory is complete. Thus we consider hypergeometric integral equations, Hankel transforms and dual integral equations of Titchmarsh type in detail. In the last chapter we mention how an incomplete theory can be developed in the case of most of the other standard transforms on $(0,\infty)$ and also indicate a number of directions in which the theory may develop in the future.

On a personal note, my interest in this field began during the period 1968-71 when it was my great pleasure and privilege to be a research student at the University of Edinburgh under the supervision of the late Professor Arthur Erdélyi. During that period and right up to his untimely death, his willing help and friendly advice were a great inspiration. Without him, this book would never have existed. I hope that it might serve as my modest tribute to a very great mathematician and friend.

It is perhaps appropriate that the impetus for me to put pen to paper came from another of Professor Erdélyi's former students, David Colton, and I am pleased to record my appreciation of Professor Colton's advice and encouragement. My thanks also go to my colleague Dr. Gary Roach for his interest and for looking through the manuscript. Last but by no means least, I would like to record my sincere thanks to Mrs. Mary Sergeant and Miss Elaine Livsey for preparing the typescript so excellently and coping with my whims and fancies.

Contents

0 Notation

Here we introduce a few standard notations and conventions which will be used throughout the book.

1. All functions will be complex-valued.
2. An expression such as x^λ, where x is a positive real number and λ is complex, will be interpreted as $\exp(\lambda \log x)$ with log x real.
3. All integrals will be Lebesgue integrals.
4. Let I denote either the open interval $(0,\infty)$ or the whole real line R^1. We consider (complex-valued) measurable functions defined almost everywhere (a.e.) on I.

   (i) f is locally integrable on I if it is (Lebesgue) integrable over every compact sub-interval of I.

   (ii) Any equation involving locally integrable functions is to be interpreted as holding a.e. on the appropriate set. Alternatively, we may work with equivalence classes of functions, two functions being in the same equivalence class if they are equal a.e.

   (iii) For $1 \leqslant p < \infty$, $L^p(I)$ is the set of (equivalence classes of) measurable functions f such that

$$||f||_p = \{\int_I |f(x)|^p dx\}^{1/p} < \infty. \tag{0.1}$$

   $L^\infty(I)$ is the set of (equivalence classes of) measurable functions f such that

$$||f||_\infty = \operatorname{ess\,sup}_I f < \infty. \qquad (0.2)$$

For $1 \leqslant p \leqslant \infty$, $L^p(I)$ is a Banach space with respect to the norm $||\ ||_p$.

(iv) Let $1 \leqslant p \leqslant \infty$ and let μ be a complex number. Then L^p_μ is the space of (equivalence classes of) functions f such that $x^{-\mu}f(x) \in L^p(0,\infty)$, i.e.,

$$L^p_\mu = \{f: x^{-\mu}f(x) \in L^p(0,\infty)\}. \qquad (0.3)$$

(We shall not require this definition on $(-\infty,\infty)$.) Occasionally we will write

$$L^p = L^p_0 = L^p(0,\infty). \qquad (0.4)$$

L^p_μ is a Banach space with respect to the norm $||\ ||_{p,\mu}$ defined by

$$||f||_{p,\mu} = ||x^{-\mu}f(x)||_p \qquad (0.5)$$

where $||\ ||_p$ is given by (0.1) or (0.2).

(v) If $1 \leqslant p \leqslant \infty$, the number q will always be related to p via the relation

$$\frac{1}{p} + \frac{1}{q} = 1 \quad \text{or} \quad q = \frac{p}{p-1} \qquad (0.6)$$

with the convention that if $p = 1$, $q = \infty$ while if $p = \infty$, $q = 1$.

5. Again, let I denote either $(0,\infty)$ or R^1. Here we consider complex-valued functions defined everywhere on I.

(i) $C^\infty(I)$ denotes the set of all (complex-valued) functions on I which are infinitely differentiable on I, i.e. which have derivatives of all orders at all points of I.

(ii) $C_0^\infty(I)$ is the subset of $C^\infty(I)$ consisting of those functions ϕ which have compact support, i.e. which are such that $\phi(x) = 0$ outside some compact subset of I (the compact subset varying with ϕ).

The set $C_0^\infty(I)$ will sometimes be denoted by $\mathfrak{D}(I)$ and, in the case $I = (0,\infty)$, simply by $\mathfrak{D}$.

6. If X is a topological vector space, we denote by X' the dual space of X, i.e. the set of all continuous linear functionals on X. The value assigned to $\phi \in X$ by $f \in X'$ will be denoted by (f,ϕ).

1 Introduction

§1.1 Motivation and Background

Let X, Y be two (non-empty) sets and T a mapping of X into Y. If $g \in Y$ does not lie in the range of T, then the equation

$$Tf = g \tag{1.1}$$

has no solution $f \in X$. Nevertheless, it is sometimes possible to recover something from the wreck.

Suppose, for instance, that it is possible to imbed the sets X and Y in sets $\tilde{X}$ and $\tilde{Y}$ respectively with $f \to \tilde{f}$, $g \to \tilde{g}$ etc. Suppose also that T can be extended to a mapping $\tilde{T}$ of $\tilde{X}$ onto $\tilde{Y}$ in such a way that, for all $f \in X$,

$$\tilde{T}\tilde{f} = \widetilde{Tf}. \tag{1.2}$$

Then, if g is as in (1.1), the equation

$$\tilde{T}h = \tilde{g} \tag{1.3}$$

will have at least one solution $h \in \tilde{X}$. Such an h might be called a generalised (or weak) solution of (1.1) since, if (1.1) has a solution $f \in X$, $h = \tilde{f}$ will satisfy (1.3) in view of (1.2). Although the word "imbed" was used above, there may not be any topologies involved initially. However if X, Y, $\tilde{X}$, $\tilde{Y}$ are topological spaces and the imbeddings are continuous, it would be ideal if $\tilde{T}$ turned out to be a homeomorphism of $\tilde{X}$ onto $\tilde{Y}$. Then we would have a unique generalised solution of (1.1).

We shall be concerned with the case when X and Y are topological vector spaces of functions (justifying the choice of f, g above) and T is an integral operator from X into Y but not onto. Indeed X and Y will be spaces of the form L^p_μ as defined in (0.3). We give two instances for the case $\mu = 0$ involving operators which will attract much of our attention in the sequel.

Example 1.1

Let η and α be complex numbers with Re $\alpha > 0$ and define $I_1^{\eta,\alpha}$ by

$$(I_1^{\eta,\alpha}f)(x) = \frac{x^{-\eta-\alpha}}{\Gamma(\alpha)}\int_0^x (x-t)^{\alpha-1}t^\eta f(t)dt \qquad (0 < x < \infty).$$

$I_1^{\eta,\alpha}$ is one of the Erdélyi-Kober operators of fractional integration introduced in [32] and [34]. Indeed in [32, Theorem 2], Kober showed that $I_1^{\eta,\alpha}$ is a continuous linear mapping of $L^p (= L^p(0,\infty))$ into itself provided that Re $\eta > -1/q$. However $I_1^{\eta,\alpha}$ does not map L^p onto L^p. For instance, if

$$I_1^{\eta,1}f = g \qquad (f,g \in L^p(0,\infty))$$

(equality holding almost everywhere on $(0,\infty)$), then $x^{\eta+1}g(x)$ must be differentiable almost everywhere on $(0,\infty)$.

Example 1.2

For suitable complex numbers ν, $1 < p \leqslant 2$ and $f \in L^p$, we may define $H_\nu f$, the Hankel transform of f of order ν, by

$$(H_\nu f)(x) = \underset{n\to\infty}{\text{l.i.m.}(q)}\int_0^n \sqrt{xt}\,J_\nu(xt)f(t)dt \qquad (0 < x < \infty)$$

where l.i.m.(q) denotes the limit in the $L^q(0,\infty)$ norm. Then by standard results in [1] and [78], H_ν is a continuous linear mapping of L^p into L^q when Re $\nu > -3/2 + 1/p$. However, except in the very special case of

$p = 2$, H_ν does not map L^p onto L^q and a useful characterisation of the range $H_\nu(L^p)$ does not seem to be known.

An even more extreme situation arises with the Laplace transform .

Example 1.3

For $f \in L^p$ $(1 \leqslant p \leqslant \infty)$, define Lf by

$$(Lf)(x) = \int_0^\infty e^{-xt} f(t)dt \qquad (0 < x < \infty).$$

Then from [85, pp. 312-3], if $1 < p < \infty$, $g = Lf$ $(f \in L^p)$ if and only if g is infinitely differentiable and, for some constant M,

$$\frac{k}{(k!)^p} \int_0^\infty \left| \frac{d^k f}{dx^k} \right|^p x^{kp+p-2} dx < M \qquad (k = 0,1,2, \ldots).$$

Thus L maps L^p into $L^p_{2/p-1}$ but clearly not onto. The cases $p = 1$ and $p = \infty$ produce analogous results.

Similar comments can be made about other standard integral transforms on the half-line $(0,\infty)$ but we have enough to be getting on with.

Returning to the general case, we have to consider how to imbed our space X of functions in a suitable larger set. One method is to take $\tilde{X} = Z'$, the dual space of some space Z of infinitely differentiable testing-functions. If Z is chosen appropriately, we might hope that each element f of X would generate a functional $\tilde{f} \in Z'$ according to the prescription

$$(\tilde{f},\phi) = \int_E f(x)\phi(x)dx \tag{1.4}$$

where, for our present discussion, E will be either $(0,\infty)$ or $(-\infty,\infty)$. Examples of such spaces Z' are the spaces $\mathfrak{D}'(-\infty,\infty)$ and $\mathfrak{D}'(0,\infty)$ of Schwartz distributions and the space $\mathcal{S}'$ of tempered distributions discussed in, for instance, [24], [75], [79] and [86]. We cannot hope for a single space Z' which will be ideal for every operator T we care to consider. Hence, in the literature, many different spaces are introduced which are tailor-made for the problem in hand. However, in the case of functions defined on $(0,\infty)$, it is usual to demand that $\mathcal{D}(0,\infty)$ be dense in Z so that the restrictions of the elements of Z' to $\mathcal{D}(0,\infty)$ form a subspace of the Schwartz space $\mathcal{D}'(0,\infty)$ by [87, Corollary 1.8 - 2a]. Similar comments apply to $(-\infty,\infty)$. Thus Z' is a space of generalised functions in the sense of Zemanian [87, p.39].

The extension of integral transformations from classical functions to generalised functions has attracted a lot of attention in recent years and mention must be made of the work of Zemanian which appears in his book [87]. Since [87] was published, further developments have taken place and we shall mention briefly a few of these below, although no attempt has been made to make the list comprehensive. We do this in the course of outlining three methods which have been successfully used to carry out the extension process.

The first method might be called "the adjoint operator method". Suppose we are dealing with two spaces X and Y of functions on $(0,\infty)$ which are imbedded in spaces Z_1', Z_2' of generalised functions respectively and let T map X into Y. Then if $f \in X$, (1.2) decrees that, for any testing-function $\phi \in Z_2$,

$$(\widetilde{Tf},\phi) = \int_0^\infty Tf(x)\phi(x)\,dx = \int_0^\infty f(x)T^*\phi(x)\,dx = (\tilde{f},T^*\phi) \tag{1.5}$$

where $T^*: Z_2 \rightarrow Z_1$ is the formal adjoint of T ([87], §1.10). (1.5) suggests that we define $\tilde{T}: Z_1' \rightarrow Z_2'$ by

$$(\tilde{T}h,\phi) = (h,T^*\phi) \qquad (h \in Z_1',\ \phi \in Z_2). \tag{1.6}$$

The properties of $\tilde{T}$ then follow from those of T* by standard theorems ([87], p. 29). Thus in this approach, we try to choose spaces Z_1 and Z_2 such that $T^*: Z_2 \to Z_1$ is a homeomorphism. Then, by [87, Theorem 1.10 - 2], $\tilde{T}: Z_1' \to Z_2'$ is also a homeomorphism. This approach is used in [87] for a study of the Hankel transform. An earlier example of the same method (on $(-\infty,\infty)$) is the now standard theory of the Fourier transform on the space $\mathcal{S}'$ of tempered distributions; see, for instance, [24], [75] and [79].

A second, more specialised method might be called "the convolution method". Again, we shall work on $(0,\infty)$ only. The method treats an integral transform T of the form

$$(Tf)(x) = \int_0^\infty k(x-t)f(t)dt \qquad (0 < x < \infty) \tag{1.7}$$

so that Tf is the convolution of the kernel k and the unknown function f. Convolution is an operation which is meaningful for distributions in $\mathcal{D}'(-\infty,\infty)$ whose support is bounded on the left ([24, Ch. 1, §5], [86, Ch. 5]) and in particular for elements of $\mathcal{D}'(0,\infty)$. Hence if k generates a distribution $\tilde{k} \in \mathcal{D}'(0,\infty)$, we are led to define $\tilde{T}$ on $\mathcal{D}'(0,\infty)$ by

$$\tilde{T}h = \tilde{k} * h \tag{1.8}$$

where * denotes distributional convolution. An example of this approach is afforded by the extension of the Riemann-Liouville fractional integral I_1^α defined for Re $\alpha > 0$ by

$$I_1^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} f(t)dt$$

so that I_1^α is one of the constituents of the operator in Example 1.1. In this case, the kernel generates the distribution $x_+^{\alpha-1}/\Gamma(\alpha)$ described in [24, p. 47] so that the extended operator, $\tilde{I}_1^\alpha$ say, is given by

$$\tilde{I}_1^{\alpha} h = x_+^{\alpha-1}/\Gamma(\alpha) * h \quad (h \in \mathcal{D}'(0,\infty)).$$

Using basic properties of convolution, a modest theory of fractional integration can be developed ([24], pp. 115 - 122).

The third method might be called the "kernel method". This is somewhat different in that it maps a generalised function into a classical function rather than another generalised function. Again, to fix ideas, consider the operator T defined by

$$(Tf)(x) = \int_0^{\infty} k(x,t)f(t)dt \quad (0 < x < \infty) \tag{1.9}$$

where k is a known kernel and $f \in X$. To imbed f in Z', we choose Z in such a way that, as a function of t, $k(x,t) \in Z$ for each fixed $x \in (0,\infty)$. Then, under appropriate conditions, $f \in X$ will generate a functional $\tilde{f} \in Z'$ and the right-hand side of (1.9) can be regarded as $(\tilde{f}, k_x)$ where

$$k_x(t) = k(x,t) \quad (0 < t < \infty). \tag{1.10}$$

This suggests that if $h \in Z'$, we take $\tilde{T}h$ to be the classical function defined by

$$\tilde{T}h(x) = (h, k_x) \quad (0 < x < \infty). \tag{1.11}$$

This method is extensively used by Zemanian in [87] where we find applications to the Laplace, Mellin, K, I and Weierstrass transforms (the K and I transforms being analogues of the Hankel transform with J_ν replaced by the modified Bessel functions K_ν and I_ν). Perhaps paradoxically, a general convolution transform is also treated by this method in [87, Chapter 8].

In recent years, the adjoint operator method or kernel method has been applied to all the standard integral transforms on $(0,\infty)$ as well as many more off-beat generalisations. There have been studies using the adjoint operator method of

fractional calculus by Erdélyi and McBride [17], [46],[47],[50], [74],

Hankel transforms by Dube and Pandey [6], Koh [35], [36] and Lee [38], [39],

Mellin, Hankel and Watson transforms and fractional integrals by Braaksma and Schuitman [2].

On the other hand, there have been studies using the kernel method of

Stieltjes transforms by Erdélyi [15], Pandey [65] and Pathak [66],

Hardy transforms by Pathak and Pandey [67].

However, as hinted above, there is a snag. In many cases the spaces of generalised functions introduced in the references quoted are expressly geared to one particular transform and seem to be of little or no use for any other transform. This is hardly surprising since the kernels of the various transforms behave so differently. Nevertheless, in all but the simplest problems, it will be necessary to apply a succession of operators in order to obtain a solution and we therefore need spaces of generalised functions relative to which all the relevant operators are well-behaved.

The object of this book is to introduce and study certain spaces of generalised functions which, we believe, are of interest as regards the L^p theory of a number of important operators on the positive half-line $(0,\infty)$. We have chosen to study a few operators in considerable detail rather than to deal sketchily with a lot of transforms. To provide some continuity, we have chosen fractional calculus as a unifying theme and we deal with a number of problems which are connected to this theme. Except in the last chapter, we use the "adjoint operator" method as described above.

Perhaps the nearest relative of our approach in the literature is that